THE CAUCHY PROBLEM FOR THE EULER-POISSON SYSTEM AND DERIVATION OF THE ZAKHAROV-KUZNETSOV EQUATION

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Abstract. We consider in this paper the rigorous justification of the Zakharov-Kuznetsov equation from the Euler-Poisson system for uniformly magnetized plasmas. We first provide a proof of the local well-posedness of the Cauchy problem for the aforementioned system in dimensions two and three. Then we prove that the long-wave small-amplitude limit is described by the Zakharov-Kuznetsov equation. This is done first in the case of cold plasma; we then show how to extend this result in presence of the isothermal pressure term with uniform estimates when this latter goes to zero.

1. Introduction

1.1. **General Setting.** The Zakharov-Kuznetsov equation

$$u_t + u\,\partial_x u + \partial_x \Delta u = 0, \qquad u = u(x,y,z,t), \qquad (x,y,z)\in\mathbb{R}^d,\ t\in\mathbb{R},\ d=2,3 \tag{1}$$

was introduced as an asymptotic model in [28] (see also [17], [15], [6], and [27] for some generalizations) to describe the propagation of nonlinear ionic-sonic waves in a magnetized plasma.

The Zakharov-Kuznetsov is a natural multi-dimensional extension of the Korteweg-de Vries equation, quite different from the well-known Kadomtsev-Petviashvili (KP) equation though.

Contrary to the Korteweg-de Vries or the Kadomtsev-Petviashvili equations, the Zakharov-Kuznetsov equation is not completely integrable but it has a hamiltonian structure and possesses two invariants, namely (for $u_0 = u(\cdot,0)$) :

$$M(t) = \int_{R^d} u^2(x,t) = \int_{R^d} u_0^2(x) = M(0) \tag{2}$$

and the hamiltonian

$$H(t) = \frac{1}{2}\int_{R^d} [|\nabla u|^2 - \frac{u^3}{3}] = \frac{1}{2}\int_{R^d} [|\nabla u_0|^2 - \frac{u_0^3}{3}] = H(0). \tag{3}$$

The Cauchy problem for the Zakharov-Kuznetsov equation has been proven to be globally well posed in the two-dimensional case for data in $H^1(\mathbb{R}^2)$ ([7]), and locally well-posed in the three-dimensional case for data in $H^s(\mathbb{R}^3)$, $s>\frac{3}{2}$, ([21]) and recently in $H^s(\mathbb{R}^3)$, $s>1$, [24]. We also refer to [22] for solutions on a nontrivial background and to [20] and [25] for well-posedness results of the Cauchy problem for *generalized* Zakharov-Kuznetsov equations in $\mathbb{R}^2$. Unique continuation properties for the Zakharov-Kuznetsov equation were established in [23], [2].

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The Zakharov-Kuznetsov equation was formally derived in [28] as a long wave small-amplitude limit of the following Euler-Poisson system in the "cold plasma" approximation,

$$\begin{cases} \partial_t n + \nabla\cdot\mathbf{v} + \nabla\cdot(n\mathbf{v}) = 0, \\ \partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\phi + a\mathbf{e}\wedge\mathbf{v} = 0, \\ \Delta\phi - e^{\phi} + 1 + n = 0. \end{cases} \tag{4}$$

Here n is the deviation of the ion density from 1, $\mathbf{v}$ is the ion velocity, ϕ the electric potential, a is a measure of the uniform magnetic field, applied along the vector $\mathbf{e} = (1,0,0)^T$ so that if $\mathbf{v} = (v_1, v_2, v_3)^T$, $\mathbf{e}\wedge\mathbf{v} = (0, -v_3, v_2)^T$.
Note that this skew-adjoint term is similar to a Coriolis term in the Euler equations for inviscid incompressible fluids.

The main goal of the present paper is to justify rigorously this formal long-wave limit. The one-dimensional case (leading to the Korteweg-de Vries equation) has been partially justified in [5], and Guo-Pu gave in a recent preprint a full justification of this limit [11].

Setting $\rho = (1+n)$, (4) writes

$$\begin{cases} \partial_t \rho + \nabla\cdot(\rho\mathbf{v}) = 0, \\ \partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\phi + a\mathbf{e}\wedge\mathbf{v} = 0, \\ \Delta\phi - e^{\phi} + \rho = 0. \end{cases} \tag{5}$$

In this formulation, the Euler-Poisson system possesses a (formally) hamiltonian conserved energy

$$\begin{aligned} H(\rho,\mathbf{v},\phi) &= \int_{\mathbb{R}^d} [\frac{1}{2}\rho|\mathbf{v}|^2 + \rho\phi - \frac{1}{2}|\nabla\phi|^2 - e^{\phi} + 1]dx \\ &= \int_{\mathbb{R}^d} [\frac{1}{2}\rho|\mathbf{v}|^2 + \frac{1}{2}|\nabla\phi|^2 + e^{\phi}(\phi - 1) + 1]dx, \end{aligned} \tag{6}$$

as it is easily seen by multiplying the first (resp. second) equation in (5) by $\frac{1}{2}|\mathbf{v}|^2 + \phi$ (resp. $\rho\mathbf{v}$), then adding and integrating over $\mathbb{R}^d$.
Of course one has to find a correct functional setting in order to justify the definition of H and its conservation (see Remark 1 in Subsection 2.1 below).
Note that the Euler-Poisson system with $a = 0$ has another formally conserved quantity, namely the impulse

$$\frac{d}{dt}\int_{\mathbb{R}^d} \rho\mathbf{v}dx = 0. \tag{7}$$

This conservation law is easily derived (formally) from the equation for $\rho\mathbf{v}$:

$$\partial_t(\rho\mathbf{v}) + \nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v}) + \rho\nabla\phi = 0. \tag{8}$$

Linearizing (4) around the constant solution $n = 1$, $\mathbf{v} = 0$, $\phi = 0$, one finds the dispersion relation for a plane wave $e^{i(\omega t - \mathbf{k}.x)}$, $\mathbf{k} = (k_1, k_2, k_3)$:

$$\omega^4(\mathbf{k}) + \omega^2(\mathbf{k})(a^2 - \frac{|\mathbf{k}|^2}{1+|\mathbf{k}|^2}) - a^2\frac{k_1^2}{1+|\mathbf{k}|^2} = 0 \tag{9}$$

or

$$(1+|\mathbf{k}|^2) - \frac{k_1^2}{\omega^2(\mathbf{k})} - \frac{|\mathbf{k}_\perp|^2}{\omega^2(\mathbf{k}) + a^2} = 0 \tag{10}$$

where $|\mathbf{k}_\perp|^2 = k_2^2 + k_3^2$.
In the absence of applied magnetic field (a =0), this relation reduces to:

$$\omega^2(\mathbf{k}) = \frac{|\mathbf{k}|^2}{1+|\mathbf{k}|^2}. \tag{11}$$

Those relations display the weakly dispersive character of the Euler-Poisson system.

Contrary to the KP case (see [1] for the justification of various asymptotic models of surface waves) the rigorous justification of the long-wave limit of the Euler-Poisson system has not been carried out so far (see however [5, 11] in the one-dimensional case). This justification is the main goal of the present paper.

To start with, we investigate the local well-posedness of the Euler-Poisson system (4) which does not raise any particular difficulty but for which, to our knowledge, no result seems to be explicitly available in the literature in dimensions 2 and 3. The paper [26] concerns a "linearized" version of (4), namely the term $\Delta\phi + 1 - e^\phi$ is replaced by its linearization at $\phi = 0$ that is $(\Delta - 1)\phi$ (see also [14] where a unique continuation property is established for the one- dimensional version of this "modified" Euler-Poisson system). Note that this "linearized" version of (4) is somewhat reminiscent of the pressureless Euler-Poisson system which has been intensively studied (see for instance [4], [3] and the references therein).

In the one-dimensional case (4) has the very simple form

$$\begin{cases} \partial_t n + [(1+n)\mathbf{v}]_x = 0, \\ \partial_t \mathbf{v} + \mathbf{v}\mathbf{v}_x + \phi_x = 0, \\ \phi_{xx} - e^\phi + 1 + n = 0. \end{cases} \tag{12}$$

The existence of supersonic solitary waves for (12) has been proven in [19]. The linear stability of those solitary waves was investigated in [13] and their interactions was studied in [12], in particular in comparison with their approximations in the long wave limit by KdV solitary waves.

Though our analysis is mainly concerned with the higher dimensional case $d = 2, 3$, our results apply as well to the system (12) and to provide an alternative proof to [11] of the justification of the KdV approximation.

Let us mention finally, that (4) is valid for cold plasmas only; in the general case, an isothermal pressure term must be added,

$$\begin{cases} \partial_t n + \nabla\cdot\mathbf{v} + \nabla\cdot(n\mathbf{v}) = 0, \\ \partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\phi + \alpha\dfrac{\nabla n}{1+n} + a\mathbf{e}\wedge\mathbf{v} = 0, \\ \Delta\phi - e^\phi + 1 + n = 0, \end{cases} \tag{13}$$

with $\alpha \geq 0$. For this system (with $\alpha > 0$), global existence for small data has been proved [10] in dimension $d = 3$ in absence of magnetic field ($a = 0$) and in the irrrotational case. Still for $\alpha > 0$, in [8], the authors provide for the full equations uniform energy estimate in the quasineutral limit (i.e. $e^\phi = 1+n$) for well prepared initial data (note that they also handle the initial boundary value problem). The derivation and justification of the KdV approximation is generalized in [11] using a different proof as in the case $\alpha = 0$. We show how to extend our results on the ZK approximation to (13) and provide uniform estimates with respect to α that allow one to handle the convergence of solutions of (13) to solutions of (4) when $\alpha \to 0$.

1.2. **Organization of the paper.** The paper is organized as follows.
In Section 2 we prove that the Cauchy problem for the Euler-Poisson system (4) is locally well-posed. The main step is to express ϕ as a function of n by solving the elliptic equation in (4) by the super and sub-solutions method. This step is of course trivial when one considers the "linearized" Euler-Poisson system. Then we establish the local well-posedness for data $(n_0, \mathbf{v}_0)$ in $H^{s-1}(\mathbb{R}^d) \times H^s(\mathbb{R}^d)^d,\ s > \frac{d}{2}+1$ such that $|n_0|_\infty < 1$, thus generalizing the result of [26]. We use in a crucial way the smoothing property of the map $n \mapsto \phi$.

In Section 3 we derive rigorously the Zakharov-Kuznetsov equation as a long wave limit of the Euler-Poisson system. In order to do this, we need to introduce a small parameter ϵ and to establish for a scaled version of the Euler-Poisson system existence and bounds on the correct time scale. However the elliptic equation for ϕ provides a smoothing effect which is not uniform with respect to ϵ and this makes the Cauchy problem more delicate (we cannot apply the previous strategy which would give an existence time shrinking to zero with ϵ). We are thus led to view the Euler-Poisson system as a semilinear perturbation of a symmetrizable quasilinear system and we have to find the correct symmetrizer. We obtain in this framework well-posedness (with uniform bounds on the correct time scale) for data in $H^s(\mathbb{R}^d) \times H^{s+1}_\epsilon(\mathbb{R}^d)^d,\ s > \frac{d}{2}+1$ (we refer to (34) for the definition of $H^{s+1}_\epsilon(\mathbb{R}^d)$). We then prove that this solution is well approximated by the solution of the ZK equation on the relevant time scales.

We finally show in Section 4 how to modify the results of Section 3 when the isothermal pressure is not neglected, i.e. when one works with (13) instead of (4). The presence of a nonzero coefficient $\alpha > 0$ in front of the isothermal pressure term induces a smoothing effect on the variable n; however, this smoothing effect vanishes as $\alpha \to 0$, and, in order to obtain an existence time uniform with respect to ϵ and α, it is necessary to work in a Banach scale indexed by these parameters and that is adapted to measure this smoothing effect.

1.3. **Notations.** - Partial differentiation are denoted by subscripts, ∂_x, ∂_t, $\partial_j = \partial_{x_j}$ etc.

- We denote by $|\cdot|_p$ $(1 \le p \le \infty)$ the standard norm of the Lebesgue spaces $L^p(\mathbb{R}^d)$ $(d = 2, 3)$.

- We use the Fourier multiplier notation: $f(D)u$ is defined as $\mathcal{F}(f(D)u)(\xi) = f(\xi)\widehat{u}(\xi)$, where $\mathcal{F}$ and $\widehat{\cdot}$ stand for the Fourier transform.

- The operator $\Lambda = (1-\Delta)^{1/2}$ is equivalently defined using the Fourier multiplier notation to be $\Lambda = (1+|D|^2)^{1/2}$.

- The standard notation $H^s(\mathbb{R}^d)$, or simply H^s if the underlying domain is clear from the context, is used for the L^2-based Sobolev spaces; their norm is written $|\cdot|_{H^s}$.

- For a given Banach space X we will denote $|.|_{X,T}$ the norm in $C([0,T]; X)$. When $X = L^p$, the corresponding norm will be denoted $|.|_{p,T}$.

- We will denote by C various absolute constants.

- The notation $A + \langle B\rangle_{s>t_0}$ refers to A if $s \le t_0$ and $A + B$ if $s > t_0$.

2. The Cauchy problem for the Euler-Poisson system

The aim of this section is to prove the local well-posedness of the Cauchy problem associated to the Euler-Poisson system.

$$\begin{cases} \partial_t n + \nabla\cdot\mathbf{v} + \nabla\cdot(n\mathbf{v}) = 0, \\ \partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\phi + a\mathbf{e}\wedge\mathbf{v} = 0, \\ \Delta\phi - e^{\phi} + 1 + n = 0 \\ n(\cdot,0) = n_0, \\ \mathbf{v}(\cdot,0) = \mathbf{v}_0. \end{cases} \tag{14}$$

2.1. **Solving the elliptic part.** We consider here, for a given n, the elliptic equation

$$L(\phi) = -\Delta\phi + e^{\phi} - 1 = n. \tag{15}$$

Proposition 1. *Let $n\in L^\infty(\mathbb{R}^d)\cap H^1(\mathbb{R}^d)$, $d=1,2,3$, such that $\inf_{\mathbb{R}^d} 1+n>0$. Then there exists a unique solution $\phi\in H^3(\mathbb{R}^d)$ of (15) such that:*
(i) The following estimate holds

$$-K_- = \ln(1-|n|_\infty) \le \phi \le K_+ = \ln(1+|n|_\infty).$$

(ii) Defining $c_\infty(n) = |(1+n)^{-1}|_\infty$ and $I_1(n) = \frac{1}{c_\infty(n)}|n|_2^2 + |n|_{H^1}^2$, one has

$$\int_{\mathbb{R}^d}[\frac{c_\infty(n)}{2}|\phi|^2 + \frac{1}{2}|\nabla\phi|^2 + |\Delta\phi|^2 + 2e^{\phi}|\nabla\phi|^2 + \frac{1}{2}(e^{\phi}-1)^2] \le \frac{1}{2}I_1(n).$$

(iii) If furthermore, $n\in H^s(\mathbb{R}^d)$, $s\ge 0$, then $\phi\in H^{s+2}(\mathbb{R}^d)$ and

$$|\Lambda^{s+1}\nabla\phi|_2 \le F_s(I_1(n),|n|_\infty)(1+|n|_{H^s}), \tag{16}$$

where $F_s(.,.)$ is an increasing function of its arguments.

Remark 1. Writing $\xi_i\xi_j\hat{\phi} = -\frac{\xi_i\xi_j}{|\xi|^2}|\xi|^2\hat{\phi}$ and using the L^2 continuity of the Riesz transforms we see that we can replace $\Delta\phi$ in the left hand side of (ii) in Proposition 1 by any derivative $\partial^\alpha\phi$, replacing possibly the right hand side by $CI_1(n)$, where C is an absolute constant.

Remark 2. Notice that by (ii), one has $|\phi|_2^2 \le \frac{I_1(n)}{c_\infty(n)}$.

Proof. We use the method of sub and super-solutions to construct the solution ϕ. As a super-solution we take $\phi_+ = K_+$ where K_+ is a positive constant satisfying

$$K_+ \ge \ln(1+|n|_\infty),$$

so that

$$L(\phi_+) \ge n.$$

As a sub-solution, we choose $\phi_- = -K_- < 0$ where K_- is a positive constant satisfying

$$K_- \ge \ln(\frac{1}{\inf_{\mathbb{R}^d}(1+n)}) = \ln(\frac{1}{c_\infty(n)}),$$

so that

$$L(\phi_-) \le n.$$

The next elementary lemma will be useful.

Lemma 1. *Let Ω be an arbitrary open subset of $\mathbb{R}^n$ and $\phi \in L^2(\Omega)\cap L^\infty(\Omega)$, $\phi \neq 0$. Then*

$$|e^\phi - 1|_2 \leq \frac{e^{|\phi|_\infty}-1}{|\phi|_\infty}|\phi|_2.$$

If moreover $\phi \in H^1(\Omega)$, then $e^\phi - 1 \in H^1(\Omega)$.

Proof. To prove the first point we expand

$$|e^\phi - 1| = |\phi \sum_{n\geq 1} \frac{\phi^{n-1}}{n!}| \leq \frac{|\phi|}{|\phi|_\infty} \sum_{n\geq 1} \frac{|\phi|_\infty^n}{n!}$$

The last assertion results from the estimate

$$|\nabla e^\phi|_2 \leq e^{|\phi|_\infty}|\nabla \phi|_2.$$

□

Let $F(\phi) = e^\phi - 1$. Then $F'(\phi)$ is bounded by a positive constant K on the interval $[-K_-, K_+]$. Let $\lambda > K$ be so that the function $\lambda I - F$ is strictly increasing in $[-K_-, K_+]$.
Let $B_p = \{x \in \mathbb{R}^d, |x| < p\}$. We consider the auxiliary problem, for a given $\phi \in H^2(B_p)$ satisfying $-K_- \leq \phi \leq K_+$,

$$\left\{\begin{array}{l} -\Delta\psi + \lambda\psi = \lambda\phi - F(\phi) + n \\ \psi_{|\partial B_p} = 0, \end{array}\right. \tag{17}$$

and we write $S(\phi) = \psi$. Since $F(\phi) \in L^2(B_p)$, $\psi = S(\phi) \in H_0^1 \cap H^2(B_p)$. Moreover, one checks easily by the maximum principle that, since $-K_- = \phi_- \leq \phi \leq K_+ = \phi_+$, then also

$$-K_- = \phi_- \leq S(\phi) \leq K_+ = \phi_+.$$

We now define inductively $\phi_0 = \phi_-$, $\phi_{k+1} = S(\phi_k)$. By the maximum principle, one checks that (ϕ_k) is increasing, $\phi_{k+1} \geq \phi_k, k \in \mathbb{N}$, and that

$$\phi_- \leq \phi_k \leq \phi_+, k \in \mathbb{N}.$$

Moreover, $S : L^2(B_p) \to L^2(B_p)$ is continuous since $z \mapsto \lambda z - F(z)$ is Lipschitz. The sequence (ϕ_k) is increasing and bounded from above. It converges almost everywhere to some ϕ which belongs to $L^2(B_p)$ since B_p is bounded. By Lebesgue's theorem, the convergence holds also in $L^2(B_p)$. On the other hand, using that F is Lipschitz on $[-K_-, K_+]$, one checks that (ϕ_k) is Cauchy in $H_0^1(B_p)$, proving that $\phi \in H_0^1(B_p)$, and is solution of (15) in B_p. Moreover $e^\phi - 1 \in L^2(B_p)$ and $\phi \in H^2(B_p)$.

Assuming that ϕ_1 and ϕ_2 are two $H_0^1 \cap H^2(B_p)$ solutions we deduce immediately that, setting $\phi = \phi_1 - \phi_2$,

$$|\nabla\phi|_2^2 + \int_{B_p} (e^{\phi_1} - e^{\phi_2})\phi = 0,$$

and we conclude that $\phi = 0$ by the monotonicity of the exponential.

To summarize, for any $p \in \mathbb{N}$, we have proven the existence of a unique solution in $H_0^1 \cap H^2(B_p)$, (which we will denote ϕ_p from now on), of the elliptic problem

$$\left\{\begin{array}{l} -\Delta\phi + e^\phi - 1 = n \\ \phi_{|\partial B_p} = 0. \end{array}\right. \tag{18}$$

Moreover, ϕ_p satisfies the bounds :

$$\phi_- = -K_- \le \phi_p \le \phi_+ = K_+. \tag{19}$$

We derive now a series of (uniform in p) estimates on ϕ_p. We first notice that for $\phi \ge -K_-$, one has with $\alpha_0 = \frac{1-e^{-K_-}}{K_-}$,

$$(e^\phi - 1)\phi \ge \alpha_0 \phi^2. \tag{20}$$

Multiplying (18) by ϕ_p and integrating over B_p we thus deduce

$$\int_{B_p} [\frac{\alpha_0}{2}|\phi_p|^2 + |\nabla\phi_p|^2] \le \frac{1}{2\alpha_0}|n|_2^2. \tag{21}$$

Multiplying (18) by $-\Delta\phi_p$ and integrating over B_p we obtain

$$\int_{Bp} |\Delta\phi_p|^2 + \int_{Bp} e^{\phi_p}|\nabla\phi_p|^2 = \int_{Bp} \nabla\phi_p \cdot \nabla n. \tag{22}$$

Finally we integrate (18) against $(e^{\phi_p} - 1)$ to get

$$\int_{B_p} e^{\phi_p}|\nabla\phi_p|^2 + \frac{1}{2}\int_{B_p} (e^{\phi_p} - 1)^2 \le \frac{1}{2}\int_{B_p} n^2. \tag{23}$$

Adding (21), (22), (23) we obtain

$$\begin{aligned}\int_{B_p} [\frac{\alpha_0}{2}|\phi_p|^2 + \frac{1}{2}|\nabla\phi_p|^2 + |\Delta\phi_p|^2 + 2e^{\phi_p}|\nabla\phi_p|^2 + \frac{1}{2}(e^{\phi_p} - 1)^2] \\ \le \frac{1}{2\alpha_0}|n|_2^2 + \frac{1}{2}|\nabla n|_2^2 + \frac{1}{2}|n|_2^2.\end{aligned} \tag{24}$$

Now we extend ϕ_p outside B_p by 0 to get a $H^1(\mathbb{R}^d)$ function $\tilde{\phi_p}$. Obviously $\tilde{\phi_p}$ satisfies the bound (21) and (23) and up to a subsequence, $\tilde{\phi_p}$ converges weakly in $H^1(\mathbb{R}^d)$, strongly in $L^2_{loc}(\mathbb{R}^d)$, and almost everywhere to some function $\phi \in H^1(\mathbb{R}^d)$ which satisfies the bound (19).

Let us prove that ϕ is solution of the elliptic equation $L\phi = n$ (see (15)). Let $\chi \in \mathfrak{D}(\mathbb{R}^d)$. Then supp $\chi \subset B_p$ for some p. Since ϕ_p solves $L\phi_p = n$ in B_p, one has

$$\int_{B_p} \nabla\phi_p \cdot \nabla\chi + \int_{B_p} (e^{\phi_p} - 1)\chi = \int_{B_p} n\chi,$$

and thus

$$\int_{\mathbb{R}^d} \nabla\tilde{\phi_p} \cdot \nabla\chi + \int_{\mathbb{R}^d} (e^{\tilde{\phi_p}} - 1)\chi = \int_{\mathbb{R}^d} n\chi.$$

One then infers by weak convergence and Lebesgue theorem (using Lemma 1) that

$$\int_{\mathbb{R}^d} \nabla\phi \cdot \nabla\chi + \int_{\mathbb{R}^d} (e^\phi - 1)\chi = \int_{\mathbb{R}^d} n\chi,$$

proving that ϕ solves

$$-\Delta\phi + e^\phi - 1 = n. \tag{25}$$

Uniqueness is derived by the same argument used for ϕ_p. By passing to the limit in (21) and (23) one obtains the corresponding estimates for ϕ. Since $e^\phi - 1 - n \in L^2(\mathbb{R}^d)$, $\phi \in H^2(\mathbb{R}^d)$ and (22) for ϕ follows. Finally the estimate (ii) in Proposition 1, that is (24) for ϕ, follows by adding the previous estimates.

We now prove the higher regularity estimates (iii) assuming that $n \in H^s(\mathbb{R}^d)$.

From the continuity of the Riesz transforms (see Remark 1), it is enough to control $|\Lambda^s \Delta\phi|_2$. We get from (25) that

$$\begin{aligned} |\Lambda^s \Delta\phi|_2 &\leq |n|_{H^s} + |e^\phi - 1|_{H^s} \\ &\leq |n|_{H^s} + C(|\phi|_\infty)|\phi|_{H^s}, \end{aligned}$$

the second line being a consequence of Moser's inequality. We can therefore deduce the result by a simple induction using the first two points of the proposition. $\square$

Remark 3. One easily checks that the energy (see (6)) makes sense for $(n, \mathbf{v}, \phi) \in H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d)^d \times L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. In fact one has then (recalling that $\rho = 1+n$),

$$\rho\phi - (e^\phi - 1) = \rho\phi - (\phi + g) = n\phi - g, \qquad g \in L^1(\mathbb{R}^d),$$

and on the other hand

$$\rho|\mathbf{v}|^2 = |\mathbf{v}|^2 + n|\mathbf{v}|^2 \in L^1(\mathbb{R}^d),$$

by Sobolev embedding.

2.2. **Local well-posedness.** We establish here the local well-posedness of the Cauchy problem for (14).

Theorem 1. *Let $s > \frac{d}{2}+1$, $n_0 \in H^{s-1}(\mathbb{R}^d)$, $\mathbf{v}_0 \in H^s(\mathbb{R}^d)^d$ such that $\inf_{\mathbb{R}^d} 1+n_0 > 0$. There exist $T > 0$ and a unique solution $(n, \mathbf{v}) \in C([0,T]; H^{s-1}(\mathbb{R}^d) \times H^s(\mathbb{R}^d)^d)$ of (14) such that $|(1+n)^{-1}|_{\infty,T} < 1$ and $\phi \in C([0,T]; H^{s+1}(\mathbb{R}^d))$.*
Moreover the energy (6) is conserved on $[0,T]$ and so is the impulse (if $a = 0$).

Proof. Solving (15) for ϕ, we set $\nabla\phi = F(n) = \nabla L^{-1}(n)$ and rewrite (14) as

$$\left\{ \begin{array}{l} \partial_t n + \nabla\cdot\mathbf{v} + \nabla\cdot(n\mathbf{v}) = 0, \\ \partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + F(n) + a\mathbf{e}\wedge\mathbf{v} = 0, \\ n(\cdot, 0) = n_0, \\ \mathbf{v}(\cdot, 0) = \mathbf{v}_0. \end{array} \right. \tag{26}$$

We derive first energy estimates. We apply the operator Λ^{s-1} to the first equation in (26), Λ^s to the second one and take the L^2 scalar product with $\Lambda^{s-1}n$ and $\Lambda^s\mathbf{v}$ respectively. Using the Kato-Ponce commutator estimates [16] and integration by parts in the first equation, we obtain (note that the skew-adjoint term does not play a role here):

$$\left\{ \begin{array}{l} \dfrac{1}{2}\dfrac{d}{dt}|\Lambda^{s-1}n|_2^2 \leq |\Lambda^s\mathbf{v}|_2(1 + |\Lambda^{s-1}n|_2) + C|\nabla\mathbf{v}|_\infty |\Lambda^{s-1}n|_2^2, \\ \dfrac{1}{2}\dfrac{d}{dt}|\Lambda^s\mathbf{v}|_2^2 \leq C|\nabla\mathbf{v}|_\infty|\Lambda^s\mathbf{v}|_2^2 + C|\Lambda^s F(n)|_2|\Lambda^s\mathbf{v}|_2. \end{array} \right. \tag{27}$$

As soon as $|(1+n)^{-1}|_{L^\infty_{(x,t)}} < \infty$, using Proposition 1 (for the existence of ϕ given n) we infer that

$$|\Lambda^s F(n)|_2 \leq F_{s-1}(I_1(n), |n|_\infty)\big(1 + |n|_{H^{s-1}}\big). \tag{28}$$

One has plainly that

$$I_1(n) \leq C|n|_{H^1}^2\Big(\frac{1}{1-|n|_\infty} + 1\Big) \leq C|n|_{H^1}^2\Big(\frac{2}{c_0} + 1\Big),$$

if $1+n \geq \frac{c_0}{2}$.
Gathering those inequalities, we obtain the system of differential inequalities for $|n(\cdot,t)|_{H^{s-1}}$ and $|\mathbf{v}(\cdot,t)|_{H^s}$, and provided that $1-|n|_\infty \geq \frac{c_0}{2}$,

$$\begin{cases} \dfrac{d}{dt}|n|_{H^{s-1}} \leq C_1(|\mathbf{n}|_{H^{s-1}}, |\mathbf{v}|_{H^s}) \\ \dfrac{d}{dt}|\mathbf{v}|_{H^s} \leq C_2(|\mathbf{n}|_{H^{s-1}}, |\mathbf{v}|_{H^s}) \end{cases} \tag{29}$$

where C_1 and C_2 are smooth functions. Let $\alpha(t) = |n(\cdot,t)|_{H^{s-1}}$ and $\beta(t) = |\mathbf{v}(\cdot,t)|_{H^s}$ and consider the differential system:

$$\begin{cases} \alpha' = C_1(\alpha,\beta)), \\ \beta' = C_2(\alpha,\beta). \end{cases} \tag{30}$$

Let (A,B) be the local solution of (30) with initial data $(A_0,B_0) = (|n_0|_{H^{s-1}}, |\mathbf{v}_0|_{H^s})$ satisfying $1+n_0 \geq c_0$. The solution to (30) exists on a time interval which length depends only on (A_0,B_0).
Coming back to (29), one deduces that $(|n(\cdot,t)|_{H^{s-1}}, |\mathbf{v}(\cdot,t)|_{H^s})$ is bounded from above by (A,B) on a time interval I which length depends only on $(|n_0|_{H^{s-1}}, |\mathbf{v}_0|_{H^s})$ (and possibly shortened to ensure that $1+n \geq \frac{c_0}{2}$ by continuity).
To complete the proof we need to smooth out (4). This can be done for instance by truncating the high frequencies, that is using $\chi(jD)$ where χ is a cut-off function and $j = 1,2,\ldots$. We obtain an ODE system in $H^{s-1} \times H^s$. The energy estimates are derived as above and one passes to the limit in a standard way.

Finally, the conservation of both the energy and the impulse (if $a = 0$) is obvious since the functional setting of Theorem 1 allows to justify their formal derivation. □

3. The long wave limit of the Euler-Poisson system

In order to justify the Zakharov-Kuznetsov equation as a long wave limit of the Euler-Poisson system with an applied magnetic field, we have to introduce an appropriate scaling.

We set $\mathbf{v} = (v_x, v_y, v_z)$. Laedke and Spatschek [17] derived formally the Zakharov-Kuznetsov equation by looking for approximate solutions of (4) of the form

$$\begin{aligned} n^\epsilon &= \epsilon\, n^{(1)}(\epsilon^{1/2}(x-t), \epsilon^{1/2}y, \epsilon^{1/2}z, \epsilon^{3/2}t) + \epsilon^2 n^{(2)} + \epsilon^3 n^{(3)} \\ \phi^\epsilon &= \epsilon\, \phi^{(1)}(\epsilon^{1/2}(x-t), \epsilon^{1/2}y, \epsilon^{1/2}z, \epsilon^{3/2}t) + \epsilon^2 \phi^{(2)} + \epsilon^3 \phi^{(3)} \\ v_x^\epsilon &= \epsilon\, v_x^{(1)}(\epsilon^{1/2}(x-t), \epsilon^{1/2}y, \epsilon^{1/2}z, \epsilon^{3/2}t) + \epsilon^2 v_x^{(2)} + \epsilon^3 v_x^{(3)} \\ v_y^\epsilon &= \epsilon^{3/2}\, v_y^{(1)}(\epsilon^{1/2}(x-t), \epsilon^{1/2}y, \epsilon^{1/2}z, \epsilon^{3/2}t) + \epsilon^2 v_y^{(2)} + \epsilon^{5/2} v_y^{(3)} \\ v_z^\epsilon &= \epsilon^{3/2}\, v_z^{(1)}(\epsilon^{1/2}(x-t), \epsilon^{1/2}y, \epsilon^{1/2}z, \epsilon^{3/2}t) + \epsilon^2 v_z^{(2)} + \epsilon^{5/2} v_z^{(3)}. \end{aligned} \tag{31}$$

The asymptotic analysis of the Euler-Poisson system is easier to handle if we work with rescaled variables and unknowns adapted to this ansatz. More precisely, if we introduce

$$\tilde{x} = \epsilon^{1/2}x, \quad \tilde{y} = \epsilon^{1/2}y, \quad \tilde{z} = \epsilon^{1/2}z, \quad \tilde{t} = \epsilon^{1/2}t, \quad \tilde{n} = \epsilon n, \quad \tilde{\phi} = \epsilon\phi, \quad \tilde{\mathbf{v}} = \epsilon\mathbf{v},$$

the Euler-Poisson equation (4) becomes (dropping the tilde superscripts),

$$\left\{\begin{array}{l}\partial_t n+\nabla\cdot((1+\epsilon n)\mathbf{v})=0,\\ \partial_t\mathbf{v}+\epsilon(\mathbf{v}\cdot\nabla)\mathbf{v}+\nabla\phi+a\epsilon^{-1/2}\mathbf{e}\wedge\mathbf{v}=0,\\ -\epsilon^2\Delta\phi+e^{\epsilon\phi}-1=\epsilon n,\end{array}\right. \tag{32}$$

and the ZK equation is derived by looking for approximate solutions to this system under the form [1]

$$\begin{aligned} n^\epsilon &= n^{(1)}(x-t,y,z,\epsilon t)+\epsilon n^{(2)}+\epsilon^2 n^{(3)}\\ \phi^\epsilon &= \phi^{(1)}(x-t,y,z,\epsilon t)+\epsilon\phi^{(2)}+\epsilon^2\phi^{(3)}\\ v_x^\epsilon &= v_x^{(1)}(x-t,y,z,\epsilon t)+\epsilon v_x^{(2)}+\epsilon^2 v_x^{(3)}\\ v_y^\epsilon &= \epsilon^{1/2}\, v_y^{(1)}(x-t,y,z,\epsilon t)+\epsilon v_y^{(2)}+\epsilon^{3/2}v_y^{(3)}\\ v_z^\epsilon &= \epsilon^{1/2}\, v_z^{(1)}(x-t,y,z,\epsilon t)+\epsilon v_z^{(2)}+\epsilon^{3/2}v_z^{(3)}.\end{aligned} \tag{33}$$

3.1. **The Cauchy problem revisited.** It is easily checked that when applied to (32), Theorem 1 provides an existence time which is of order $O(1)$ with respect to ϵ, while the time scale $O(1/\epsilon)$ must be reached to observe the dynamics of the Zakharov-Kuznetsov equation that occur along the slow time scale ϵt.
As explained in the Introduction, we therefore need, in order to justify the Zakharov-Kuznetsov equation as a long wave limit of the Euler-Poisson system, to solve the Cauchy problem associated to (32) on a time interval of order $O(1/\epsilon)$. In order to do so, we will consider (32) as a perturbation of a hyperbolic quasilinear system and give a proof which does not use the smoothing effect of the ϕ equation for a fixed ϵ. This is the reason why more regularity is required on the initial data in the statement of the theorem below. Before stating it, let us introduce the space H^{s+1}_ϵ which is the standard Sobolev space $H^{s+1}(\mathbb{R}^d)$ endowed with the norm

$$\forall s\geq 0,\quad \forall f\in H^{s+1},\quad |f|^2_{H^{s+1}_\epsilon}=|f|^2_{H^s}+\epsilon|\nabla f|^2_{H^s}; \tag{34}$$

the presence of the small parameter ϵ in front of the second term is here to make this norm adapted to measure the smoothing effects of the ϕ equation; the fact that these smoothing effects are small explains why Theorem 1, which relies on them, provides an existence time much too small to observe the dynamics of the Zakharov-Kuznetsov equation.

Theorem 2. *Let $s>\frac{d}{2}+1$ and $n_0\in H^s(\mathbb{R}^d)$, $\mathbf{v}_0\in H^{s+1}(\mathbb{R}^d)^d$ such that $1+n_0\geq c_0$ for some $c_0>0$.*
Then there exists $\underline{T}>0$ such that for all $\epsilon\in(0,1)$, there is a unique solution $(n^\epsilon,\mathbf{v}^\epsilon,\phi^\epsilon)\in C([0,\frac{\underline{T}}{\epsilon}];H^s(\mathbb{R}^d)\times H^{s+1}_\epsilon(\mathbb{R}^d)^d\times H^{s+1}(\mathbb{R}^d))$ of (32) such that $1+\epsilon n^\epsilon>c_0/2$.
Moreover the family $(n^\epsilon,\mathbf{v}^\epsilon,\nabla\phi^\epsilon)_{\epsilon\in(0,1)}$ is uniformly bounded in $H^s\times H^{s+1}_\epsilon\times H^{s-1}$.

Proof. **Step 1. Preliminary results.** If we differentiate the third equation of (32) with respect to ∂_j $(j=x,y,z,t)$, we get

$$M_\epsilon(\phi)\partial_j\phi=\partial_j n\quad\text{ with }\quad M_\epsilon(\phi)=-\epsilon\Delta+e^{\epsilon\phi}; \tag{35}$$

the operator $M_\epsilon(\phi)$ plays a central role in the energy estimates below, and we state here some basic estimates. The first is that $(u,M_\epsilon(\phi)u)^{1/2}$ defines a norm which is

[1] Actually we will not use the third order profiles.

uniformly equivalent to $|\cdot|_{H^1_\epsilon}$; more precisely, for all $u\in H^1(\mathbb{R}^d)$,

$$(u, M_\epsilon(\phi)u) \leq C(|\phi|_\infty)|u|^2_{H^1_\epsilon} \quad \text{and} \quad |u|^2_{H^1_\epsilon} \leq C(|\phi|_\infty)(u, M_\epsilon(\phi)u). \tag{36}$$

We also have for all $u\in H^1(\mathbb{R}^d)$ and $f\in W^{1,\infty}(\mathbb{R}^d)$,

$$\big(u, fM_\epsilon(\phi)u\big) \leq C(|\phi|_\infty, |f|_\infty, \sqrt{\epsilon}|\nabla f|_\infty)\, |u|^2_{H^1_\epsilon},$$

and, for all $u\in H^1(\mathbb{R}^d)$ and $f\in W^{2,\infty}(\mathbb{R}^d)$,

$$\big(u, [f\partial_j, M_\epsilon(\phi)]u\big) \leq C(|\phi|_{W^{1,\infty}}, |f|_{W^{1,\infty}}, \sqrt{\epsilon}|\nabla\partial_j f|_\infty)\, |u|^2_{H^1_\epsilon} \qquad (j=x,y,z);$$

these two estimates are readily obtained from the definition of $M_\epsilon(\phi)$ and integration by parts.

Let us finally prove that $M_\epsilon(\phi)$ is invertible and give estimates on its inverse.

Lemma 2. *Let $\phi\in L^\infty(\mathbb{R}^d)$. Then $M_\epsilon(\phi): H^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ is an isomorphism and*

$$\forall v\in L^2(\mathbb{R}^d), \qquad e^{-\frac{\epsilon}{2}|\phi|_\infty}|M_\epsilon(\phi)^{-1}v|_2 + \sqrt{\epsilon}|\nabla M_\epsilon(\phi)^{-1}v|_2 \leq e^{\frac{\epsilon}{2}|\phi|_\infty}|v|_2.$$

If moreover $t_0>d/2$, $s\geq 0$ and $\phi\in H^{t_0+1}\cap H^s(\mathbb{R}^d)$, then we also have

$$\forall v\in H^s(\mathbb{R}^d), \qquad |M_\epsilon(\phi)^{-1}v|_{H^s} + \sqrt{\epsilon}|\nabla M_\epsilon(\phi)^{-1}v|_{H^s} \leq C(|\phi|_{H^{t_0+1}\cap H^s})|v|_{H^s}.$$

Proof of the lemma. The invertibility property of $M_\epsilon(\phi)$ follows classically from Lax-Milgram's theorem, and the first estimate of the lemma follows from the coercivity property of $M_\epsilon(\phi)$, namely,

$$e^{-\epsilon|\phi|_\infty}|u|_2^2 + \epsilon|\nabla u|_2^2 \leq (M_\epsilon(\phi)u, u). \tag{37}$$

In order to prove the higher order estimates, let us write $u=M_\epsilon(\phi)^{-1}v$. We have by definition $(-\epsilon\Delta + e^{\epsilon\phi})u = v$, so that applying Λ^s on both sides, we get

$$(-\epsilon\Delta + e^{\epsilon\phi})\Lambda^s u = \Lambda^s v - [\Lambda^s, e^{\epsilon\phi}]u. \tag{38}$$

Using the first estimate, and recalling that $u=M_\epsilon(\phi)^{-1}v$, we deduce that

$$|M_\epsilon(\phi)^{-1}v|_{H^s} + \sqrt{\epsilon}|\nabla M_\epsilon(\phi)^{-1}v|_{H^s} \leq e^{\epsilon|\phi|_\infty}\big(|v|_{H^s} + |[\Lambda^s, e^{\epsilon\phi}]M_\epsilon(\phi)^{-1}v|_2\big).$$

Now, the use Kato-Ponce and Coifman-Meyer commutator estimates, and Moser's inequality, shows that

$$\begin{aligned}|[\Lambda^s, e^{\epsilon\phi}]M_\epsilon(\phi)^{-1}v|_2 &\leq \epsilon C(|\phi|_\infty)\\ &\times \Big(|\phi|_{H^{t_0+1}}|M_\epsilon(\phi)^{-1}v|_{H^{s-1}} + \langle|\phi|_{H^s}|M_\epsilon(\phi)^{-1}v|_{H^{t_0}}\rangle_{s>t_0+1}\Big),\end{aligned}$$

and we get therefore

$$\begin{aligned}|M_\epsilon(\phi)^{-1}v|_{H^s} + \sqrt{\epsilon}|\nabla M_\epsilon(\phi)^{-1}u|_{H^s} \quad\leq\quad & C(|\phi|_\infty)\big(|v|_{H^s} + |\phi|_{H^{t_0+1}}|M_\epsilon(\phi)^{-1}v|_{H^{s-1}}\\ &+ \langle|\phi|_{H^s}|M_\epsilon(\phi)^{-1}v|_{H^{t_0}}\rangle_{s>t_0+1}\big),\end{aligned}$$

and the results follows by a continuous induction. $\square$

Step 2. L^2 estimates for a linearized system. Let $T>0$ and $(\underline{n},\underline{\mathbf{v}})\in L^\infty([0,T];W^{1,\infty}(\mathbb{R}^d))\cap W^{1,\infty}([0,T];L^\infty(\mathbb{R}^d))$ be such that $1+\epsilon\underline{n}\geq c_0>0$ uniformly on $[0,T]$, and $\underline{\phi}\in W^{1,\infty}$ with $\partial_t\underline{\phi}\in L^\infty$. Let us consider a classical solution $(n,\mathbf{v},\phi)$ of the linear system

$$\begin{cases}\partial_t n + (1+\epsilon\underline{n})\nabla\cdot\mathbf{v} + \epsilon\underline{\mathbf{v}}\cdot\nabla n = \epsilon f,\\ \partial_t\mathbf{v} + \epsilon(\underline{\mathbf{v}}\cdot\nabla)\mathbf{v} + \nabla\phi + a\epsilon^{-1/2}\mathbf{e}\wedge\mathbf{v} = \epsilon\mathbf{g},\\ M_\epsilon(\underline{\phi})\nabla\phi = \nabla n + \epsilon\mathbf{h}.\end{cases} \tag{39}$$

We want to prove here that

$$\sup_{[0,T]}\big(|n|_2^2+|\mathbf{v}|_{H^1_\epsilon}^2\big) \leq \exp(\epsilon C_0 T) \times \big(|n_{|t=0}|_2^2+|\mathbf{v}_{|t=0}|^2_{H^1_\epsilon}\big)+\epsilon T(|f|^2_{L^2_T}+|\mathbf{g}|^2_{H^1_{\epsilon,T}}+|\mathbf{h}|^2_{L^2_T})\big), \tag{40}$$

with $C_0=C(\frac{1}{c_0},|(\underline{n},\underline{\mathbf{v}},\phi)|_{W^{1,\infty}_T},|\partial_t(\underline{n},\underline{\mathbf{v}},\underline{\phi})|_{L^\infty_T},\sqrt{\epsilon}|\nabla\nabla\cdot\underline{\mathbf{v}}|_{L^\infty_T})$.
Taking the L^2-scalar product with $\frac{1}{1+\epsilon\underline{n}}n$, we get

$$\big(\frac{1}{1+\epsilon\underline{n}}\partial_t n,n\big)+\big(\nabla\cdot\mathbf{v},n\big)+\epsilon\big(\frac{\underline{\mathbf{v}}}{1+\epsilon\underline{n}}\cdot\nabla n,n\big)=\epsilon\big(\frac{f}{1+\epsilon\underline{n}},n\big),$$

which can be rewritten under the form

$$\frac12\partial_t\big(\frac{1}{1+\epsilon\underline{n}}n,n\big)+\epsilon\frac12\big(\frac{\partial_t\underline{n}}{(1+\epsilon\underline{n})^2}n,n\big)+(\nabla\cdot\mathbf{v},n) -\epsilon\frac12\big(n,\nabla\cdot(\frac{\underline{\mathbf{v}}}{1+\epsilon\underline{n}})n\big)=\epsilon\big(\frac{f}{1+\epsilon\underline{n}},n\big). \tag{41}$$

We now take the scalar product of the second equation with $M_\epsilon(\underline{\phi})\mathbf{v}$ to obtain, after recalling that $M_\epsilon(\underline{\phi})\nabla\phi=\nabla n+\epsilon\mathbf{h}$,

$$\begin{aligned}\big(M_\epsilon(\underline{\phi})\partial_t\mathbf{v},\mathbf{v}\big)+\epsilon\big(M_\epsilon(\underline{\phi})\underline{\mathbf{v}}\cdot\nabla\mathbf{v},\mathbf{v}\big)+\big(\nabla n,\mathbf{v}\big)&\\ =-\epsilon\big(\mathbf{h},\mathbf{v}\big)+\epsilon\big(M_\epsilon(\underline{\phi})\mathbf{g},\mathbf{v}\big),&\end{aligned}$$

or equivalently

$$\begin{aligned}&\frac12\partial_t\big(M_\epsilon(\underline{\phi})\mathbf{v},\mathbf{v}\big)-\frac12\epsilon\big(\partial_t\underline{\phi}e^{\epsilon\underline{\phi}}\mathbf{v},\mathbf{v}\big)-\frac12\epsilon\big(\mathbf{v},(\nabla\cdot\underline{\mathbf{v}})M_\epsilon(\underline{\phi})\mathbf{v}\big)\\ &-\frac12\epsilon\big(\mathbf{v},[\underline{\mathbf{v}}\cdot\nabla,M_\epsilon(\underline{\phi})]\mathbf{v}\big)+(\nabla n,\mathbf{v})=-\epsilon\big(\mathbf{h},\mathbf{v}\big)+\epsilon\big(M_\epsilon(\underline{\phi})\mathbf{g},\mathbf{v}\big).\end{aligned} \tag{42}$$

Adding (42) to (41), we get therefore

$$\begin{aligned}\frac12\partial_t\big(\frac{1}{1+\epsilon\underline{n}}n,n\big)+\frac12\partial_t\big(M_\epsilon(\underline{\phi})\mathbf{v},\mathbf{v}\big)&=\epsilon\frac12\big([\frac{\partial_t\underline{n}}{(1+\epsilon\underline{n})^2}+\nabla\cdot(\frac{\underline{\mathbf{v}}}{1+\epsilon\underline{n}})]n,n\big)\\ &+\frac12\epsilon\big(\partial_t\underline{\phi}e^{\epsilon\underline{\phi}}\mathbf{v},\mathbf{v}\big)+\frac12\epsilon\big(\mathbf{v},(\nabla\cdot\underline{\mathbf{v}})M_\epsilon(\underline{\phi})\mathbf{v}\big)+\frac12\epsilon\big(\mathbf{v},[\underline{\mathbf{v}}\cdot\nabla,M_\epsilon(\underline{\phi})]\mathbf{v}\big)\\ &+\epsilon\big(\frac{1}{1+\epsilon\underline{n}}f,n\big)-\epsilon\big(\mathbf{h},\mathbf{v}\big)+\epsilon\big(M_\epsilon(\underline{\phi})\mathbf{g},\mathbf{v}\big).\end{aligned}$$

All the terms on the right-hand-side are easily controlled (with the help of Step 1 for third and fourth terms) to obtain

$$\begin{aligned}\partial_t\big\{\big(\frac{n}{1+\epsilon\underline{n}},n\big)+\big(M_\epsilon(\underline{\phi})\mathbf{v},\mathbf{v}\big)\big\}\leq \epsilon C(\frac{1}{c_0},|(\underline{n},\underline{\mathbf{v}},\underline{\phi})|_{W^{1,\infty}},|\partial_t(\underline{n},\underline{\mathbf{v}},\underline{\phi})|_\infty,\epsilon|\nabla\nabla\cdot\underline{\mathbf{v}}|_2)&\\ \times\big(\big(\frac{1}{1+\epsilon\underline{n}}n,n\big)+|\mathbf{v}|^2_{H^1_\epsilon}+|f|_2^2+|\mathbf{g}|^2_{H^1_\epsilon}+|\mathbf{h}|_2^2\big).&\end{aligned}$$

Using (36) and a Gronwall inequality, we readily deduce (40).
Step 3. **H^s estimates for a linearized system.** We want to prove here that the solution $(n,\mathbf{v},\phi)$ to (39) satisfies, for all $s\geq t_0+1$,

$$\begin{aligned}\sup_{[0,T]}\big(|n|^2_{H^s}+|\mathbf{v}|^2_{H^{s+1}_\epsilon}\big) \leq \exp(\epsilon C_s T)\times\Big(&|n_{|t=0}|^2_{H^s}+|\mathbf{v}_{|t=0}|^2_{H^{s+1}_\epsilon}\big)\\ &+\epsilon T(|f|^2_{H^s_T}+|\mathbf{g}|^2_{H^{s+1}_{\epsilon,T}}+|\mathbf{h}|^2_{H^s_T}+|n|^2_{H^s}+|\mathbf{v}|^2_{H^{s+1}_\epsilon})\Big),\end{aligned} \tag{43}$$

with $C_s = C(\frac{1}{c_0}, |\underline{n}|_{H^s_T}, |\underline{\mathbf{v}}|_{H^{s+1}_{\epsilon,T}}, |\underline{\phi}|_{H^s_T}, |\partial_t(\underline{n}, \underline{\mathbf{v}}, \underline{\phi})|_{L^\infty_T})$.
Applying Λ^s to the three equations of (39), and writing $\tilde{n} = \Lambda^s n$, $\tilde{\mathbf{v}} = \Lambda^s \mathbf{v}$, and $\tilde{\phi} = \Lambda^s \phi$, we get

$$\left\{ \begin{array}{l} \partial_t \tilde{n} + (1+\epsilon \underline{n})\nabla \cdot \tilde{\mathbf{v}} + \epsilon \underline{\mathbf{v}} \cdot \nabla \tilde{n} = \epsilon \tilde{f}, \\ \partial_t \tilde{\mathbf{v}} + \epsilon(\underline{\mathbf{v}} \cdot \nabla)\tilde{\mathbf{v}} + \nabla \tilde{\phi} + a\epsilon^{-1/2}\mathbf{e} \wedge \tilde{\mathbf{v}} = \epsilon \tilde{\mathbf{g}}, \\ M_\epsilon(\underline{\phi})\nabla \tilde{\phi} = \nabla \tilde{n} + \epsilon \tilde{\mathbf{h}}. \end{array} \right. \tag{44}$$

with

$$\begin{array}{rcl} \tilde{f} &=& \Lambda^s f - [\Lambda^s, \underline{n}]\nabla \cdot \mathbf{v} - [\Lambda^s, \underline{\mathbf{v}}] \cdot \nabla n, \\ \tilde{\mathbf{g}} &=& \Lambda^s \mathbf{g} - [\Lambda^s, \underline{\mathbf{v}}] \cdot \nabla \mathbf{v}, \\ \tilde{\mathbf{h}} &=& \Lambda^s \mathbf{h} - \dfrac{1}{\epsilon}[\Lambda^s, M_\epsilon(\underline{\phi})]\nabla \phi. \end{array}$$

From Step 2, we get that

$$\sup_{[0,T]} \big(|\tilde{n}|_2^2 + |\tilde{\mathbf{v}}|^2_{H^1_\epsilon}\big) \leq \exp(\epsilon C_0 T)$$
$$\times \big(|\tilde{n}_{|t=0}|_2^2 + |\tilde{\mathbf{v}}_{|t=0}|^2_{H^1_\epsilon}\big) + \epsilon T (|\tilde{f}|^2_{L^2_T} + |\tilde{\mathbf{g}}|^2_{H^1_{\epsilon,T}} + |\tilde{\mathbf{h}}|^2_{L^2_T})\big).$$

Now, by standard commutator estimates, we have for all $s \geq t_0 + 1$,

$$|\tilde{f}|_2 + |\tilde{\mathbf{g}}|_{H^1_\epsilon} \leq |f|_{H^s} + |\mathbf{g}|_{H^{s+1}_\epsilon} + \big(|\underline{n}|_{H^s} + |\underline{\mathbf{v}}|_{H^{s+1}_\epsilon}\big) \times \big(|\mathbf{v}|_{H^{s+1}_\epsilon} + |n|_{H^s}\big)$$

and

$$\begin{array}{rcl} |\tilde{\mathbf{h}}|_2 &\leq& |\mathbf{h}|_{H^s} + C(|\underline{\phi}|_{H^s})|\nabla \phi|_{H^{s-1}}, \\ &\leq& C(|\underline{\phi}|_{H^s})\big(|\mathbf{h}|_{H^s} + |n|_{H^s}\big), \end{array}$$

where we used Lemma 2 to get the last inequality. We can now directly deduce (43) from the Sobolev embedding $W^{1,\infty}(\mathbb{R}^d) \subset H^s(\mathbb{R}^d)$ $(s \geq t_0 + 1)$.
Step 4. End of the proof. The exact solution provided by Theorem 1 solves (39) with $(\underline{n}, \underline{\mathbf{v}}, \underline{\phi}) = (n, \mathbf{v}, \phi)$ and $f = 0$, $\mathbf{g} = 0$, $\mathbf{h} = 0$; we deduce therefore from Step 3 that it satisfies the estimate

$$\sup_{[0,T]} \big(|n|^2_{H^s} + |\mathbf{v}|^2_{H^{s+1}_\epsilon}\big) \leq \exp(\epsilon C_s T) \times \Big(|n_{|t=0}|^2_{H^s} + |\mathbf{v}_{|t=0}|^2_{H^{s+1}_\epsilon}\big) + \epsilon T (|n|^2_{H^s} + |\mathbf{v}|^2_{H^{s+1}_\epsilon})\Big),$$

with

$$\begin{array}{rcl} C_s &=& C(\dfrac{1}{c_0}, |n|_{H^s_T}, |\mathbf{v}|_{H^{s+1}_{\epsilon,T}}, |\phi|_{H^s}, |\partial_t(n, \mathbf{v}, \phi)|_{L^\infty_T}) \\ &=& C(\dfrac{1}{c_0}, |n|_{H^s_T}, |\mathbf{v}|_{H^{s+1}_{\epsilon,T}}), \end{array}$$

where we used (35) and the equation to control the time derivatives in terms of space derivatives, and Proposition 1 to control the L^2-norm of ϕ (control on $\nabla \phi$ being provided by Lemma 2). This provides us with a $\underline{T} > 0$ such that $|n|^2_{H^s} + |\mathbf{v}|^2_{H^{s+1}_\epsilon}$ remains uniformly bounded with respect to ϵ on $[0, \underline{T}/\epsilon]$. $\square$

3.2. **The ZK approximation to the Euler-Poisson system.** We construct here an approximate solution to (32) based on the ZK equation. Following Laedke and Spatschek [19], but with the rescaled variables and unknowns introduced in (31), we look for approximate solutions of (32) under the form (33), namely,

$$
\begin{aligned}
n^\epsilon &= n^{(1)}(x-t,y,z,\epsilon t)+\epsilon n^{(2)}\\
\phi^\epsilon &= \phi^{(1)}(x-t,y,z,\epsilon t)+\epsilon \phi^{(2)}\\
v_x^\epsilon &= v_x^{(1)}(x-t,y,z,\epsilon t)+\epsilon v_x^{(2)}\\
v_y^\epsilon &=\epsilon^{1/2}\, v_y^{(1)}(x-t,y,z,\epsilon t)+\epsilon v_y^{(2)}\\
v_z^\epsilon &=\epsilon^{1/2}\, v_z^{(1)}(x-t,y,z,\epsilon t)+\epsilon v_z^{(2)}.
\end{aligned}
\tag{45}
$$

Notation 1. We will denote by X the variable $x-t$ and by T the slow time variable ϵt.

Plugging this ansatz into the first equation of (32) we obtain:

$$
\partial_t n^\epsilon+\nabla\cdot\big(1+\epsilon n^\epsilon\big)\mathbf{v}^\epsilon=\sum_{j=0}^{6}\epsilon^{j/2}N^j,
\tag{46}
$$

where

$$
\begin{aligned}
N^0 &= -\frac{\partial}{\partial X}n^{(1)}+\frac{\partial}{\partial X}v_x^{(1)}\\
N^1 &= \frac{\partial}{\partial y}v_y^{(1)}+\frac{\partial}{\partial z}v_z^{(1)}\\
N^2 &= \frac{\partial}{\partial T}n^{(1)}-\frac{\partial}{\partial X}n^{(2)}+\frac{\partial}{\partial X}(n^{(1)}v_x^{(1)})+\frac{\partial}{\partial X}v_x^{(2)}+\frac{\partial}{\partial y}v_y^{(2)}+\frac{\partial}{\partial z}v_z^{(2)}\\
N^3 &= \frac{\partial}{\partial y}(n^{(1)}v_y^{(1)})+\frac{\partial}{\partial z}(n^{(1)}v_z^{(1)})\\
N^4 &= \frac{\partial}{\partial T}n^{(2)}+\frac{\partial}{\partial X}(n^{(1)}v_x^{(2)})+\frac{\partial}{\partial y}(n^{(1)}v_y^{(2)})\\
&\quad+\frac{\partial}{\partial z}(n^{(1)}v_z^{(2)})+\frac{\partial}{\partial X}(n^{(2)}v_x^{(1)}),\\
N^5 &= \frac{\partial}{\partial y}(n^{(2)}v_y^{(1)})+\frac{\partial}{\partial z}(n^{(2)}v_z^{(1)}),\\
N^6 &= \frac{\partial}{\partial X}(n^{(2)}v_x^{(2)})+\frac{\partial}{\partial y}(n^{(2)}v_y^{(2)})+\frac{\partial}{\partial z}(n^{(2)}v_z^{(2)}).
\end{aligned}
$$

Similarly, one has

$$
\frac{\partial}{\partial t}v_x^\epsilon+\epsilon v_x^\epsilon\frac{\partial}{\partial x}v_x^\epsilon+\epsilon v_y^\epsilon\frac{\partial}{\partial y}v_x^\epsilon+\epsilon v_z^\epsilon\frac{\partial}{\partial z}v_x^\epsilon+\frac{\partial}{\partial x}\phi^\epsilon=\sum_{j=0}^{6}\epsilon^{j/2}R_j^1
\tag{47}
$$

with $R_1^1 = 0$ and

$$
\begin{aligned}
R_0^1 &= -\frac{\partial}{\partial X} v_x^{(1)} + \frac{\partial}{\partial X}\phi^{(1)} \\
R_2^1 &= \frac{\partial}{\partial T} v_x^{(1)} - \frac{\partial}{\partial X} v_x^{(2)} + v_x^{(1)} \frac{\partial}{\partial X} v_x^{(1)} + \frac{\partial}{\partial X}\phi^{(2)} \\
R_3^1 &= v_y^{(1)} \frac{\partial}{\partial y} v_x^{(1)} + v_z^{(1)} \frac{\partial}{\partial z} v_x^{(1)} \\
R_4^1 &= \frac{\partial}{\partial T} v_x^{(2)} + \frac{\partial}{\partial X}(v_x^{(1)} v_x^{(2)}) + v_y^{(2)} \frac{\partial}{\partial y} v_x^{(1)} + v_z^{(2)} \frac{\partial}{\partial z} v_x^{(1)} \\
R_5^1 &= v_y^{(1)} \frac{\partial}{\partial y} v_x^{(2)} + v_z^{(1)} \frac{\partial}{\partial z} v_x^{(2)} \\
R_6^1 &= v_y^{(2)} \frac{\partial}{\partial y} v_x^{(2)} + v_z^{(2)} \frac{\partial}{\partial z} v_x^{(2)} + v_x^{(2)} \frac{\partial}{\partial X} v_x^{(2)};
\end{aligned}
$$

for the second component of the velocity equation, we get

$$
\frac{\partial}{\partial t} v_y^\epsilon + \epsilon v_x^\epsilon \frac{\partial}{\partial x} v_y^\epsilon + \epsilon v_y^\epsilon \frac{\partial}{\partial y} v_y^\epsilon + \epsilon v_z^\epsilon \frac{\partial}{\partial z} v_y^\epsilon + \frac{\partial}{\partial y}\phi^\epsilon - a\epsilon^{-1/2}\, v_z^\epsilon = \sum_{j=0}^{6} \epsilon^{j/2} R_j^2 \tag{48}
$$

with

$$
\begin{aligned}
R_0^2 &= \frac{\partial}{\partial y}\phi^{(1)} - a v_z^{(1)} \\
R_1^2 &= -(\frac{\partial}{\partial X} v_y^{(1)} + a v_z^{(2)}) \\
R_2^2 &= -\frac{\partial}{\partial X} v_y^{(2)} + \frac{\partial}{\partial y}\phi^{(2)} \\
R_3^2 &= \frac{\partial}{\partial T} v_y^{(1)} + v_x^{(1)} \frac{\partial}{\partial X} v_y^{(1)} \\
R_4^2 &= \frac{\partial}{\partial T} v_y^{(2)} + v_x^{(1)} \frac{\partial}{\partial X} v_y^{(2)} + v_y^{(1)} \frac{\partial}{\partial y} v_y^{(1)} + v_z^{(1)} \frac{\partial}{\partial z} v_y^{(1)} \\
R_5^2 &= v_x^{(2)} \frac{\partial}{\partial X} v_y^{(1)} + \frac{\partial}{\partial y}(v_y^{(1)} v_y^{(2)}) + v_z^{(1)} \frac{\partial}{\partial z} v_y^{(2)} + v_z^{(2)} \frac{\partial}{\partial z} v_y^{(1)} \\
R_6^2 &= v_x^{(2)} \frac{\partial}{\partial X} v_y^{(2)} + v_y^{(2)} \frac{\partial}{\partial y} v_y^{(2)} + v_z^{(2)} \frac{\partial}{\partial z} v_y^{(2)},
\end{aligned}
$$

while for the third component, the equations are

$$
\frac{\partial}{\partial t} v_z^\epsilon + \epsilon v_x^\epsilon \frac{\partial}{\partial x} v_z^\epsilon + \epsilon v_y^\epsilon \frac{\partial}{\partial y} v_z^\epsilon + \epsilon v_z^\epsilon \frac{\partial}{\partial z} v_z^\epsilon + \frac{\partial}{\partial z}\phi^\epsilon + a\epsilon^{-1/2}\, v_y^\epsilon = \sum_{j=0}^{6} \epsilon^{j/2} R_j^3 \tag{49}
$$

with

$$\begin{aligned}
R_0^3 &= \frac{\partial}{\partial z}\phi^{(1)} + av_y^{(1)}\\
R_1^3 &= -\frac{\partial}{\partial X}v_z^{(1)} + av_y^{(2)}\\
R_2^3 &= -\frac{\partial}{\partial X}v_z^{(2)} + \frac{\partial}{\partial z}\phi^{(2)}\\
R_3^3 &= \frac{\partial}{\partial T}v_z^{(1)} + v_x^{(1)}\frac{\partial}{\partial X}v_z^{(1)}\\
R_4^3 &= \frac{\partial}{\partial T}v_z^{(2)} + v_x^{(1)}\frac{\partial}{\partial X}v_z^{(2)} + v_y^{(1)}\frac{\partial}{\partial y}v_z^{(1)} + v_z^{(1)}\frac{\partial}{\partial z}v_z^{(1)}\\
R_5^3 &= v_x^{(2)}\frac{\partial}{\partial X}v_z^{(1)} + \frac{\partial}{\partial z}(v_z^{(1)}v_z(2)) + v_y^{(1)}\frac{\partial}{\partial y}v_z^{(2)} + v_y^{(2)}\frac{\partial}{\partial y}v_z^{(1)}\\
R_6^3 &= v_x^{(2)}\frac{\partial}{\partial X}v_z^{(2)} + v_y^{(2)}\frac{\partial}{\partial y}v_z^{(2)} + v_z^{(2)}\frac{\partial}{\partial z}v_z^{(2)}.
\end{aligned}$$

Finally, for the equation on the potential, we obtain

$$-\epsilon^2\Delta\phi^\epsilon + e^{\epsilon\phi} - 1 - \epsilon n = \epsilon r^2 + \epsilon^2 r^4 + \epsilon^3 r^6 + O(\epsilon^4) \tag{50}$$

with

$$\begin{aligned}
r^2 &= \phi^{(1)} - n^{(1)}\\
r^4 &= -\Delta\phi^{(1)} + \phi^{(2)} + \frac{1}{2}(\phi^{(1)})^2 - n^{(2)}\\
r^6 &= -\Delta\phi^{(2)} + \phi^{(1)}\phi^{(2)} + \frac{1}{6}(\phi^{(1)})^3.
\end{aligned}$$

We first derive (following essentially [19]) the equations corresponding to the successive cancellation of the leading order remainder terms; we then show that they imply that $n^{(1)}$ must solve the Zakharov-Kuznetsov equation, and then turn to show in the spirit of Ben-Youssef and Colin who considered the one-dimensional case in [5] (see also [11]) that it is indeed possible to construct an approximate solution (45) satisfying all the cancellation conditions previously derived. The consequence is the consistency property of (45) stated in Proposition 2.

3.2.1. *Cancellation of terms of order zero in ϵ.* Canceling the terms N^0 and R_0^j $(j = 1, 2, 3)$ is equivalent to the following conditions,

$$v_x^{(1)} = \phi^{(1)} = n^{(1)} \qquad \text{(assuming that } v_x^{(1)},\ \phi^{(1)},\ n^{(1)} \text{ vanish as } |X| \to +\infty) \tag{51}$$

$$v_y^{(1)} = -\frac{1}{a}\partial_z n^{(1)} \tag{52}$$

$$v_z^{(1)} = \frac{1}{a}\partial_y n^{(1)}. \tag{53}$$

3.2.2. *Cancellation of terms of order $\epsilon^{1/2}$.* Using (52)-(53), the cancellation of the terms N^1 and R_1^j $(j = 1, 2, 3)$ is equivalent to

$$v_y^{(2)} = \frac{1}{a^2}\partial^2_{Xy}n^{(1)} \quad \text{and} \quad v_z^{(2)} = \frac{1}{a^2}\partial^2_{Xz}n^{(1)}. \tag{54}$$

3.2.3. *Cancellation of terms of order ϵ.* Using the conditions derived above, the cancellation of N^2 and R_2^j $(j = 1, 2, 3)$,

$$\partial_T n^{(1)} + 2n^{(1)}\partial_X n^{(1)} + \frac{1}{a^2}\partial_X \Delta n^{(1)} = -\partial_X (v_x^{(2)} - n^{(2)}) \tag{55}$$

$$\partial_X (v_x^{(2)} - \phi^{(2)}) = \partial_T n^{(1)} + n^{(1)}\partial_X n^{(1)} \tag{56}$$

$$\partial_y \phi^{(2)} = \frac{1}{a^2}\partial^3_{XXy} n^{(1)} \tag{57}$$

$$\partial_z \phi^{(2)} = \frac{1}{a^2}\partial^3_{XXz} n^{(1)} \tag{58}$$

while the cancellation of r_2 is equivalent to $\phi^{(1)} = n^{(1)}$ which has already been imposed.

3.2.4. *Cancellation of terms of order $\epsilon^{3/2}$.* It is possible to cancel the terms of order $\epsilon^{3/2}$ for the equation on the density and on the first component of the velocity; the fact that $N^3 = R_3^1 = 0$ is actually a direct consequence of (51)-(53) and(54). Looking for the cancellation of the other components of the velocity equation, namely, setting $R_3^2 = R_3^3 = 0$ yields respectively

$$0 = -\frac{1}{a}[\partial^2_{zT} n^{(1)} + n^{(1)}\partial^2_{zX} n^{(1)}],$$
$$0 = \frac{1}{a}[\partial^2_{yT} n^{(1)} + n^{(1)}\partial^2_{yX} n^{(1)}] \ ,$$

which are inconsistent with the other equations on $n^{(1)}$; consequently, *we cannot expect a better error than $O(\epsilon^{3/2})$ on the equations for the transverse components of the velocity.*

3.2.5. *Cancellation of terms of order ϵ^2.* In order to justify the ZK approximation, we need to cancel the $O(\epsilon^2)$ terms in the equation for ϕ^ϵ, that is, to impose $r^4 = 0$, leading to the equation

$$\phi^{(2)} - n^{(2)} = \Delta n^{(1)} - \frac{1}{2}(n^{(1)})^2. \tag{59}$$

3.2.6. *Derivation of the Zakharov-Kuznetsov equation.* Combining (55), (56) and (59), we find that $n^{(1)}$ must solve the Zakharov-Kuznetsov equation,

$$2\partial_T n^{(1)} + 2n^{(1)}\partial_X n^{(1)} + (\frac{1}{a^2} + 1)\Delta\partial_X n^{(1)} = 0. \tag{60}$$

3.2.7. *Construction of the profiles.* The ZK equation being locally well posed on $H^s(\mathbb{R}^d)$, for all $s > d/2 + 1$, [2] we can consider a solution $n^{(1)} \in C([0, T_0]; H^s(\mathbb{R}^d))$, $s \geq 5$ to (60) for some $T_0 > 0$. We show here how to construct all the quantities involved in (45) in terms of $n^{(1)}$.

- In agreement with (51), we set $\phi^{(1)} = v_x^{(1)} = n^{(1)}$.
- Equations (52)-(53) then give $v_y^{(1)}, v_z^{(1)} \in C([0, T_0]; H^{s-1}(\mathbb{R}^d))$.
- We then use (54) to obtain $v_y^{(2)}, v_z^{(2)} \in C([0, T_0]; H^{s-2}(\mathbb{R}^d))$.
- Taking $\phi^{(2)} = \frac{1}{a^2}\partial^2_{XX} n^{(1)} \in C([0, T_0]; H^{s-2}(\mathbb{R}^d))$ then ensures that (57) and (58) hold.
- We then get the density corrector $n^{(2)} \in C([0, T_0]; H^{s-2}(\mathbb{R}^d))$ by (59).

[2] See [7, 21, 24] for the Cauchy theory in larger spaces.

- We recover $v_x^{(2)} \in C([0,T_0];H^{s-2}(\mathbb{R}^d))$ from (55) or (56) — the fact that $n^{(1)}$ solves the ZK equation ensures that we find the same expression with (55) or (56) .

The computations above and the explicit expression of the remaining residual terms in (46), (47), (48), (49) and (50) imply the following consistency result.

Proposition 2. *Let $T_0 > 0$, $n_0 \in H^s(\mathbb{R}^d)$ $(s \geq 5)$ and $n^{(1)} \in C([0,T_0];H^s(\mathbb{R}^d))$, solving*

$$2\partial_T n^{(1)} + 2n^{(1)}\partial_X n^{(1)} + \left(1 + \frac{1}{a^2}\right)\Delta\partial_X n^{(1)} = 0, \qquad n^{(1)}_{|_{t=0}} = n_0.$$

Constructing the other profiles as indicated above, the approximate solution $(n^\epsilon, \mathbf{v}^\epsilon, \phi^\epsilon)$ given by (45) *solves (32) up to order ϵ^3 in ϕ^ϵ, ϵ^2 in n^ϵ, v_x^ϵ, and up to order $\epsilon^{3/2}$ in v_y^ϵ, v_z^ϵ:*

$$\begin{cases} \partial_t n^\epsilon + \nabla\cdot((1+\epsilon n^\epsilon)\mathbf{v}^\epsilon) = \epsilon^2 N^\epsilon, \\ \partial_t \mathbf{v}^\epsilon + \epsilon(\mathbf{v}^\epsilon\cdot\nabla)\mathbf{v}^\epsilon + \nabla\phi^\epsilon + a\epsilon^{-1/2}\mathbf{e}\wedge\mathbf{v}^\epsilon = \epsilon^{3/2}R^\epsilon, \\ -\epsilon^2\Delta\phi^\epsilon + e^{\epsilon\phi^\epsilon} - 1 = \epsilon n^\epsilon + \epsilon^3 r^\epsilon, \end{cases} \tag{61}$$

with $R^\epsilon = (\epsilon^{1/2}R_1^\epsilon, R_2^\epsilon, R_3^\epsilon)$ and

$$|N^\epsilon|_{L^\infty([0,\frac{T_0}{\epsilon}];H^{s-5})} + \sum_{j=1}^3 |R_j^\epsilon|_{L^\infty([0,\frac{T_0}{\epsilon}];H^{s-5})} + |r^\epsilon|_{L^\infty([0,\frac{T_0}{\epsilon}];H^{s-4})} \leq C\big(T_0, |n_0|_{H^s}\big).$$

3.3. **Justification of the Zakharov-Kuznetsov approximation.**

We are now set to justify the Zakharov-Kuznetsov approximation.

Theorem 3. *Let $n^0 \in H^{s+5}$, with $s > d/2+1$, such that $1 + n^0 \geq c_0$ on $\mathbb{R}^d$ for some constant $c_0 > 0$. There exists $T_1 > 0$ such that for all $\epsilon \in (0,1)$,*
i. *The Zakharov-Kuznetsov approximation $(n^\epsilon, \mathbf{v}^\epsilon, \phi^\epsilon)$ of Proposition 2 exists on the time interval $[0, T_1/\epsilon]$;*
ii. *There exists a unique solution $(\underline{n}, \underline{\mathbf{v}}, \underline{\phi}) \in C([0,\frac{T_1}{\epsilon}];H^s(\mathbb{R}^d)\times H_\epsilon^{s+1}(\mathbb{R}^d)^d \times H^{s+1}(\mathbb{R}^d))$ provided by Theorem 2 to the Euler-Poisson equations* (32) *with initial condition $(\underline{n}^0, \underline{\mathbf{v}}^0, \underline{\phi}^0) = (n^\epsilon, \mathbf{v}^\epsilon, \phi^\epsilon)_{|_{t=0}}$.*
Moreover, one has the error estimate

$$\forall 0 \leq t \leq T_1/\epsilon, \qquad |\underline{n}(t) - n^\epsilon(t)|_{H^s}^2 + |\underline{\mathbf{v}}(t) - \mathbf{v}^\epsilon(t)|_{H_\epsilon^{s+1}}^2 \leq \epsilon^{3/2} t C(\frac{1}{c_0}, T_1, |n^0|_{H^{s+5}}).$$

Remark 4. The error of the approximation is $O(\epsilon^{3/2})$ for times of order $O(1)$ but of size $O(\epsilon^{1/2})$ for times of order $O(1/\epsilon)$. Looking at (45), this is a relative error of size $O(\epsilon^{1/2})$ for n and v_x but of size $O(1)$ for v_y and v_z. Consequently, the Zakharov-Kuznetsov approximation provides a good approximation for the density and the longitudinal velocity, but not, for large times, for the transverse velocity (at least, we did not prove it in Theorem 3).

Remark 5. In the one dimensional case (KdV approximation), all the terms of order $\epsilon^{3/2}$ can be cancelled (see §3.2.4) and the residual in the second equation of (84) is of size $O(\epsilon^2)$ instead of $O(\epsilon^{3/2})$. The error in the theorem then becomes $O(\epsilon^2 t)$, which gives a relative error of size $O(\epsilon)$ for both the density and the velocity over large times $O(1/\epsilon)$. In the one dimensional case, it is also possible to construct higher order approximations by including the order three and higher terms in the ansatz (33). This has been done in [11].

Remark 6. In [11], the authors justify the KdV approximation (corresponding to the one dimensional ZK approximation) by looking at an exact solution as a perturbation of the approximate solution, $(n_{ex}, v_{ex}) = (n_{app}, v_{app}) + \epsilon^k(n_R, v_R)$, (with $k > 0$ depending on the order of the approximation). They study the equations satisfied by (n_R, v_R), which requires subtle estimates. Our approach is much simpler: we prove uniform (with respect to ϵ) well posedness of the Euler Poisson equation, from which we deduce very easily that any consistent approximation remains close to the exact solution (but of course, in a lower norm).

Proof. Let us take $0 < T_1 \leq \min\{\underline{T}, T_0\}$, where $\underline{T}/\epsilon$ is the existence time of the exact solution provided by Theorem 2, and T_0/ϵ the existence time of the approximate solution in Proposition 2. Denote by $(\underline{n}, \underline{\mathbf{v}}, \underline{\phi}) \in C([0, \frac{T_1}{\epsilon}]; H^s(\mathbb{R}^d) \times H^{s+1}_\epsilon(\mathbb{R}^d)^d \times H^{s+1}(\mathbb{R}^d)$ the exact solution to (32) with the same initial conditions as $(n^\epsilon, \mathbf{v}^\epsilon, \phi^\epsilon)$ furnished by Theorem 2. We also write

$$(n, \mathbf{v}, \phi) = (n^\epsilon, \mathbf{v}^\epsilon, \phi^\epsilon) - (\underline{n}, \underline{v}, \underline{\phi}).$$

Taking the difference between (84) and (32), we get

$$\begin{cases} \partial_t n + (1 + \epsilon\underline{n})\nabla\cdot\mathbf{v} + \epsilon\underline{\mathbf{v}}\cdot\nabla n + \epsilon\nabla n^\epsilon\cdot\mathbf{v} + \epsilon(\nabla\cdot\mathbf{v}^\epsilon)n = \epsilon f, \\ \partial_t\mathbf{v} + \epsilon(\underline{\mathbf{v}}\cdot\nabla)\mathbf{v} + \epsilon\mathbf{v}\cdot\nabla\mathbf{v}^\epsilon + \nabla\phi + a\epsilon^{-1/2}\mathbf{e}\wedge\mathbf{v} = \epsilon\mathbf{g}, \\ M_\epsilon(\underline{\phi})\nabla\phi = \nabla n - \epsilon\phi e^{\epsilon\phi^\epsilon}\nabla\phi^\epsilon + \epsilon\mathbf{h}. \end{cases} \tag{62}$$

with $f = \epsilon N^\epsilon$, $\mathbf{g} = \epsilon^{1/2}R^\epsilon$ and $\mathbf{h} = \epsilon\nabla r^\epsilon + \big(\dfrac{e^{\epsilon\underline{\phi}} - e^{\epsilon\phi^\epsilon}}{\epsilon} - (\underline{\phi} - \phi^\epsilon)e^{\epsilon\phi^\epsilon}\big)\nabla\phi^\epsilon$. This system is of the form (39) with additional linear terms (namely, $\epsilon\nabla n^\epsilon\cdot\mathbf{v} + \epsilon(\nabla\cdot\mathbf{v}^\epsilon)n$ in the first equation, $\epsilon\mathbf{v}\cdot\nabla\mathbf{v}^\epsilon$ in the second one, and $-\epsilon\phi e^{\epsilon\phi^\epsilon}\nabla\phi^\epsilon$ in the third one) that do not affect the derivation of the energy estimate (43). The only difference is that the constant C_s in (43) must also depend on $|(n^\epsilon, \mathbf{v}^\epsilon, \phi^\epsilon)|_{H^{s+1}_T}$. Since the initial conditions for (62) are identically zero, this yields, for all $0 \leq T \leq T_1/\epsilon$,

$$\begin{aligned} \sup_{[0,T]}\big(|n|^2_{H^s} + |\mathbf{v}|^2_{H^{s+1}_\epsilon}\big) &\leq \exp(\epsilon\widetilde{C}_s T) \\ &\quad\times\epsilon T\big(|f|^2_{H^s_T} + |\mathbf{g}|^2_{H^{s+1}_{\epsilon,T}} + |\mathbf{h}|^2_{H^s_T} + |n|^2_{H^s} + |\mathbf{v}|^2_{H^{s+1}_\epsilon}\big), \end{aligned} \tag{63}$$

with $\widetilde{C}_s = C\big(\frac{1}{c_0}, |\underline{n}|_{H^s_T}, |\underline{\mathbf{v}}|_{H^{s+1}_{\epsilon,T}}, |\underline{\phi}|_{H^s_T}, |\partial_t(\underline{n}, \underline{\mathbf{v}}, \underline{\phi})|_{L^\infty_T}, |(n^\epsilon, \mathbf{v}^\epsilon, \phi^\epsilon)|_{H^{s+1}_T}\big)$. Taking if necessary a smaller $T_1 > 0$, this implies that for all $0 \leq t \leq T_1/\epsilon$,

$$|n(t)|^2_{H^s} + |\mathbf{v}(t)|^2_{H^{s+1}_\epsilon} \leq \epsilon t\exp(\widetilde{C}_s T_1) \times \big(|f|^2_{H^s_t} + |\mathbf{g}|^2_{H^{s+1}_{\epsilon,t}} + |\mathbf{h}|^2_{H^s_t}\big). \tag{64}$$

Now, one the one hand, we have

$$\begin{aligned} \widetilde{C}_s &= C\big(\frac{1}{c_0}, |\underline{n}|_{H^s_T}, |\underline{\mathbf{v}}|_{H^{s+1}_{\epsilon,T}}, |\underline{\phi}|_{H^s_T}, |\partial_t(\underline{n}, \underline{\mathbf{v}}, \underline{\phi})|_{L^\infty_T}, |(n^\epsilon, \mathbf{v}^\epsilon, \phi^\epsilon)|_{H^{s+1}_T}\big) \\ &= C\big(\frac{1}{n_0}, |n^0|_{H^{s+5}}\big), \end{aligned}$$

where we used Theorem 2 to control the norms of $(\underline{n}, \underline{\mathbf{v}}, \underline{\phi})$ in terms of $(n^0, \mathbf{v}^0)$ (with $\mathbf{v}^0$ given in terms of n^0 by $\mathbf{v}^0 = \mathbf{v}^\epsilon_{|t=0}$), and the expression of all the components of $(n^\epsilon, \mathbf{v}^\epsilon, \phi^\epsilon)$ in terms of $n^{(1)}$ to control the norms of the approximate solution. On the other hand, we get from the definition of f, $\mathbf{g}$, $\mathbf{h}$ and Proposition 2 that

$$|f|^2_{H^s_t} + |\mathbf{g}|^2_{H^{s+1}_{\epsilon,t}} + |\mathbf{h}|^2_{H^s_t} \leq \epsilon^{1/2}C(T_1, |n^0|_{H^{s+5}}).$$

We deduce therefore from (64) that

$$\forall 0 \leq t \leq T_1/\epsilon, \qquad |n(t)|^2_{H^s} + |\mathbf{v}(t)|^2_{H^{s+1}_\epsilon} \leq \epsilon^{3/2} t C(\frac{1}{c_0}, T_1, |n^0|_{H^{s+5}}).$$

□

4. The Euler-Poisson system with isothermal pressure

In [11], the authors derived and justified a version of the KdV (and therefore $d=1$) equation in the case where the isothermal pressure is not neglected. In the general of dimension $d \geq 1$, the equations (4) are then given by

$$\begin{cases} \partial_t n + \nabla\cdot\mathbf{v} + \nabla\cdot(n\mathbf{v}) = 0, \\ \partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\phi + \alpha\dfrac{\nabla n}{1+n} + a\mathbf{e}\wedge\mathbf{v} = 0, \\ \Delta\phi - e^{\phi} + 1 + n = 0, \end{cases} \tag{65}$$

where α is a positive constant related to the ratio of the ion temperature over the ion mass. In the case of cold plasmas considered in the previous sections, one has $\alpha = 0$. In [11], the cases $\alpha = 0$ and $\alpha > 0$ are treated differently, and the limit $\alpha \to 0$ (or, for instance, $\alpha = O(\epsilon)$) cannot be handled. We show here that the proof of Theorem 2 can easily be adapted to the general case $\alpha \geq 0$, hereby allowing the limit $\alpha \to 0$ and providing a generalization of the results of [11] to the case $d \geq 1$. We first extend Theorem 2 to the general Euler-Poisson system with isothermal pressure (65). We then indicate how to derive and justify a generalization of the Zakharov-Kuznetsov approximation taking into account this new term, in the same spirit as the KdV approximation derived in the one-dimensional case in [11].

4.1. **The Cauchy problem for the Euler-Poisson system with isothermal pressure.** As in Section 3, we work with rescaled equations. More precisely, we perform the same rescaling as for (32); without the "cold plasma " assumption, this system must be replaced by

$$\begin{cases} \partial_t n + \nabla\cdot((1+\epsilon n)\mathbf{v}) = 0, \\ \partial_t \mathbf{v} + \epsilon(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\phi + \alpha\dfrac{\nabla n}{1+\epsilon n} + a\epsilon^{-1/2}\mathbf{e}\wedge\mathbf{v} = 0, \\ -\epsilon^2\Delta\phi + e^{\epsilon\phi} - 1 = \epsilon n, \end{cases} \tag{66}$$

with $\alpha \geq 0$. The presence of the extra term $\alpha\frac{\nabla n}{1+\epsilon n}$ in the second equation induces a smoothing effect that allows the authors of [11] to use the pseudo-differential estimates of Grenier [9]. However, these smoothing effects disappear when $\alpha \to 0$, and the existence time thus obtained is not uniform with respect to α. We provide here a generalization of Theorem 2 that gives a uniform existence time with respect to ϵ *and* α (so that solutions to (66) provided by Theorem 4 can be seen as limits when $\alpha \to 0$ of solutions to (66)). This evanescent smoothing effect is taken into account by working with $n \in H^{s+1}_{\epsilon\alpha}(\mathbb{R}^d)$ rather than $n \in H^s(\mathbb{R}^d)$ as in Theorem 2 (from the definition (34) of $H^s_{\epsilon\alpha}$, $H^{s+1}_{\epsilon\alpha}(\mathbb{R}^d)$ coincides with $H^s(\mathbb{R}^d)$ when $\alpha = 0$).

Theorem 4. *Let $s > \frac{d}{2}+1$, $\alpha_0 > 0$ and $n_0 \in H^{s+1}_{\epsilon\alpha_0}(\mathbb{R}^d)$, $\mathbf{v}_0 \in H^{s+1}(\mathbb{R}^d)^d$ such that $1 - |n_0|_\infty \geq c_0$ for some $c_0 > 0$.*
Then there exist $\underline{T} > 0$ such that for all $\epsilon \in (0,1)$ and $\alpha \in (0,\alpha_0)$, there is a unique solution $(n^{\epsilon,\alpha}, \mathbf{v}^{\epsilon,\alpha}) \in C([0,\frac{\underline{T}}{\epsilon}]; H^{s+1}_{\epsilon\alpha}(\mathbb{R}^d)\times H^{s+1}_\epsilon(\mathbb{R}^d)^d)$ of (32) such that $1+\epsilon n > c_0/2$ and $\phi^{\epsilon,\alpha} \in C([0,T]; H^{s+1}(\mathbb{R}^d))$.

Moreover the family $(n^{\epsilon,\alpha},\mathbf{v}^{\epsilon,\alpha},\nabla\phi^{\epsilon,\alpha})_{\epsilon\in(0,1),\alpha\in(0,\alpha_0)}$ *is uniformly bounded in* $H^{s+1}_{\epsilon\alpha}\times H^{s+1}_{\epsilon}\times H^{s-1}$.

Proof. The proof follows the same steps as the proof of Theorem 2; in addition to the operator $M_\epsilon(\phi)$ defined in (35), we also need to define another second order self-adjoint operator $N_{\epsilon,\alpha}(\phi,n)$ as

$$N_{\epsilon,\alpha}(\phi,n)=\frac{1}{1+n}+\frac{1}{1+n}M_\epsilon(\phi)\frac{1}{1+n}, \tag{67}$$

provided that $\inf_{\mathbb{R}^d}(1+n)>0$.

Step 1. Preliminary results. The operator $N_{\epsilon,\alpha}(\phi,n)$ defined in (67) inherit from the properties of $M_\epsilon(\phi)$ the following estimates that echoe (36),

$$\begin{aligned}
&\big(u,N_{\epsilon,\alpha}(\phi,n)v\big)\le C\big(|\phi|_\infty,|n|_\infty\big)\big|\frac{u}{1+\epsilon n}\big|_{H^1_{\epsilon\alpha}}\big|\frac{v}{1+\epsilon n}\big|_{H^1_{\epsilon\alpha}},\\
&\big|\frac{u}{1+\epsilon n}\big|^2_{H^1_{\epsilon\alpha}}\le C\big(|\phi|_\infty,|n|_\infty\big)\big(u,N_{\epsilon,\alpha}(\phi,n)u\big).
\end{aligned}$$

We also have the following commutator estimates, that are similar to those satisfied by $M_\epsilon(\phi)$ (see Step 1 in the proof of Theorem 2),

$$\begin{aligned}
\big(u,[\partial_t,N_{\epsilon,\alpha}(\phi,n)]u\big)&\le \epsilon C\big(\frac{1}{c_0},|\partial_t n|_\infty,|n|_{W^{1,\infty}},|\partial_t\phi|_\infty,|\phi|_\infty\big)\big|\frac{u}{1+\epsilon n}\big|^2_{H^1_{\epsilon\alpha}}\\
\big(u,[f\partial_j,N_{\epsilon,\alpha}(\phi,n)]u\big)&\le C\big(\frac{1}{c_0},|(n,\phi,f)|_{W^{1,\infty}},\sqrt{\epsilon\alpha}|\nabla\partial_j n|_\infty,\sqrt{\epsilon}|\nabla\partial_j f|_\infty\big)\\
&\qquad\times\big|\frac{u}{1+\epsilon n}\big|^2_{H^1_{\epsilon\alpha}}.
\end{aligned}$$

Step 2. L^2 estimates for a linearized system. Without the “cold plasma” approximation, one must replace (39) by

$$\begin{cases}
\partial_t n+(1+\epsilon\underline{n})\nabla\cdot\mathbf{v}+\epsilon\underline{\mathbf{v}}\cdot\nabla n=\epsilon f,\\
\partial_t\mathbf{v}+\epsilon(\underline{\mathbf{v}}\cdot\nabla)\mathbf{v}+\nabla\phi+\alpha\dfrac{\nabla n}{1+\epsilon\underline{n}}+a\epsilon^{-1/2}\mathbf{e}\wedge\mathbf{v}=\epsilon\mathbf{g},\\
M_\epsilon(\underline{\phi})\nabla\phi=\nabla n+\epsilon\mathbf{h}.
\end{cases} \tag{68}$$

The presence of the new term in the second equation of (68) yields some smoothing effect on the estimate on n that are absent when $\alpha=0$; this smoothing is measured with the $H^1_{\epsilon\alpha}(\mathbb{R}^d)$ norm, which coincides with the L^2 norm used for (68) when $\alpha=0$. More precisely, we want to prove here that

$$\begin{aligned}
&\sup_{[0,T]}\big(|n|^2_{H^1_{\epsilon\alpha}}+|\mathbf{v}|^2_{H^1_\epsilon}\big)\le\exp(\epsilon C_0T)\\
&\times\big(|n_{|t=0}|^2_{H^1_{\epsilon\alpha}}+|\mathbf{v}_{|t=0}|^2_{H^1_\epsilon}\big)+\epsilon T(|f|^2_{H^1_{\epsilon\alpha,T}}+|\mathbf{g}|^2_{H^1_{\epsilon,T}}+|\mathbf{h}|^2_{L^2_T})\big),
\end{aligned} \tag{69}$$

with $C_0=C(\frac{1}{c_0},|(\underline{n},\underline{\mathbf{v}},\phi)|_{W^{1,\infty}_T},|\partial_t(\underline{n},\underline{\mathbf{v}},\underline{\phi})|_{L^\infty_T},\sqrt{\epsilon}|\underline{\mathbf{v}}|_{W^{2,\infty}_T},\sqrt{\epsilon\alpha}|n|_{W^{2,\infty}_T})$.
Instead of multiplying the first equation of (68) by $(1+\epsilon\underline{n})^{-1}$ as in the case $\alpha=0$, we multiply it by $N_{\epsilon,\alpha}(\underline{\phi},\underline{n})$ to obtain

$$\begin{aligned}
\big(N_{\epsilon,\alpha}(\underline{\phi},\underline{n})\partial_t n,n\big)+\big([1+\alpha\frac{1}{1+\epsilon\underline{n}}M_\epsilon(\underline{\phi})]\nabla\cdot\mathbf{v},n\big)+\epsilon\big(N_{\epsilon,\alpha}(\underline{\phi},\underline{n})\mathbf{v}\cdot\nabla n,n\big)&\\
=\epsilon\big(N_{\epsilon,\alpha}(\underline{\phi},\underline{n})f,n\big)&,
\end{aligned}$$

which can be rewritten under the form

$$\frac{1}{2}\partial_t\big(N_{\epsilon,\alpha}(\underline{\phi},\underline{n})n,n\big)-\frac{1}{2}\big([\partial_t,N_{\epsilon,\alpha}(\underline{\phi},\underline{n})]n,n\big)+\big([1+\alpha\frac{1}{1+\epsilon\underline{n}}M_\epsilon(\underline{\phi})]\nabla\cdot\mathbf{v},n\big)$$
$$-\epsilon\frac{1}{2}\big(n,[\underline{\mathbf{v}}\cdot\nabla,N_{\epsilon,\alpha}(\underline{\phi},\underline{n})]n\big)=\epsilon\big(N_{\epsilon,\alpha}(\underline{\phi},\underline{n})f,n\big). \tag{70}$$

As in the proof of Theorem 2, we take the L^2 scalar product of the second equation with $M_\epsilon(\phi)\mathbf{v}$; after remarking that

$$M_\epsilon(\underline{\phi})\big(\nabla\phi+\alpha\frac{\nabla n}{1+\epsilon\underline{n}}\big)=[1+\alpha M_\epsilon(\underline{\phi})\frac{1}{1+\epsilon\underline{n}}]\nabla n+\epsilon\mathbf{h}$$

we obtain with the same computations the following generalization of (42),

$$\frac{1}{2}\partial_t\big(M_\epsilon(\underline{\phi})\mathbf{v},\mathbf{v}\big)-\frac{1}{2}\epsilon\big(\partial_t\underline{\phi}e^{\epsilon\underline{\phi}}\mathbf{v},\mathbf{v}\big)-\frac{1}{2}\epsilon\big(\mathbf{v},(\nabla\cdot\underline{\mathbf{v}})M_\epsilon(\underline{\phi})\mathbf{v}\big)$$
$$-\frac{1}{2}\epsilon\big(\mathbf{v},[\underline{\mathbf{v}}\cdot\nabla,M_\epsilon(\underline{\phi})]\mathbf{v}\big)+\big([1+\alpha M_\epsilon(\underline{\phi})\frac{1}{1+\epsilon\underline{n}}]\nabla n,\mathbf{v}\big)$$
$$=-\epsilon\big(\mathbf{h},\mathbf{v}\big)+\epsilon\big(M_\epsilon(\underline{\phi})\mathbf{g},\mathbf{v}\big). \tag{71}$$

Adding (71) to (70), and proceeding exactly as in the proof of Theorem 2, we get (69).

Step 3. H^s estimates for a linearized system. We want to prove here that for all $s\geq 0$, the solution $(n,\mathbf{v},\phi)$ to (68) satisfies, for all $s\geq t_0+1$,

$$\sup_{[0,T]}\big(|n|^2_{H^{s+1}_{\epsilon\alpha}}+|\mathbf{v}|^2_{H^{s+1}_\epsilon}\big)\quad\leq\quad\exp(\epsilon C_sT)\times\Big(|n_{|t=0}|^2_{H^{s+1}_{\epsilon\alpha}}+|\mathbf{v}_{|t=0}|^2_{H^{s+1}_\epsilon}\big)$$
$$+\epsilon T(|f|^2_{H^{s+1}_{\epsilon\alpha,T}}+|\mathbf{g}|^2_{H^{s+1}_{\epsilon,T}}+|\mathbf{h}|^2_{H^s_T}+|n|^2_{H^{s+1}_{\epsilon\alpha}}+|\mathbf{v}|^2_{H^{s+1}_\epsilon})\Big), \tag{72}$$

with $C_s=C(\frac{1}{c_0},|\underline{n}|_{H^{s+1}_{\epsilon\alpha,T}},|\underline{\mathbf{v}}|_{H^{s+1}_{\epsilon,T}},|\underline{\phi}|_{H^s_T},|\partial_t(\underline{n},\underline{\mathbf{v}},\underline{\phi})|_{H^s_T})$.

Applying Λ^s to the three equations of (39), and writing $\tilde{n}=\Lambda^s n$, $\tilde{\mathbf{v}}=\Lambda^s\mathbf{v}$, and $\tilde{\phi}=\Lambda^s\phi$, we get

$$\begin{cases}\partial_t\tilde{n}+(1+\epsilon\underline{n})\nabla\cdot\tilde{\mathbf{v}}+\epsilon\underline{\mathbf{v}}\cdot\nabla\tilde{n}=\epsilon\tilde{f},\\ \partial_t\tilde{\mathbf{v}}+\epsilon(\underline{\mathbf{v}}\cdot\nabla)\tilde{\mathbf{v}}+\nabla\tilde{\phi}+\alpha\dfrac{\nabla\tilde{n}}{1+\epsilon\underline{n}}+a\epsilon^{-1/2}\mathbf{e}\wedge\tilde{\mathbf{v}}=\epsilon\tilde{\mathbf{g}},\\ M_\epsilon(\underline{\phi})\nabla\tilde{\phi}=\nabla\tilde{n}+\epsilon\tilde{\mathbf{h}}.\end{cases} \tag{73}$$

with $\tilde{f}$ and $\tilde{\mathbf{h}}$ as in the proof of Theorem 2, while $\tilde{\mathbf{g}}$ must be changed into $\tilde{\mathbf{g}}=\tilde{\mathbf{g}}+\alpha[\Lambda^s,\frac{n}{1+\epsilon\underline{n}}]\nabla\tilde{n}$. Using Step 2, one can mimic the proof of the Theorem 2. The only new ingredients needed are a control in $H^1_{\epsilon\alpha}(\mathbb{R}^d)$ of $\alpha[\Lambda^s,\frac{n}{1+\epsilon\underline{n}}]\nabla\tilde{n}$ (the new term in $\tilde{\mathbf{g}}$) and a control of $\tilde{f}$ in $H^1_{\epsilon\alpha}(\mathbb{R}^d)$ instead of $L^2(\mathbb{R}^d)$. Classical commutator estimates (see for instance [18]) yield for $s>d/2+1$,

$$\big|\alpha[\Lambda^s,\frac{n}{1+\epsilon\underline{n}}]\nabla\tilde{n}\big|_{H^1_\epsilon}\leq C(\frac{1}{c_0},|\underline{n}|_{H^{s+1}_{\epsilon\alpha}})|\tilde{n}|_{H^{s+1}_{\epsilon\alpha}},$$
$$|\tilde{f}|_{H^1_{\epsilon\alpha}}\leq|f|_{H^{s+1}_{\epsilon\alpha}}+\big(|\underline{n}|_{H^{s+1}_{\epsilon\alpha}}+|\underline{\mathbf{v}}|_{H^{s+1}_\epsilon}\big)\times\big(|\mathbf{v}|_{H^{s+1}_\epsilon}+|n|_{H^{s+1}_{\epsilon\alpha}}\big),$$

so that (72) follows exactly as (43).

Step 4. End of the proof. The end of the proof is exactly similar the same as for Theorem 2 (the exact solution is no longer furnished by Theorem 1 but by a standard iterative scheme). $\square$

Remark 7. The proof given in Theorem 1 for the case $\alpha = 0$ does not work when $\alpha > 0$.

4.2. **Derivation of a Zakharov-Kuznetsov equation in presence of isothermal pressure.** We proceed similarly to the cold plasma case but we replace the ansatz (45) by

$$\begin{aligned} n^\epsilon &= n^{(1)}(x-ct,y,z,\epsilon t) + \epsilon n^{(2)} \\ \phi^\epsilon &= \phi^{(1)}(x-ct,y,z,\epsilon t) + \epsilon \phi^{(2)} \\ v_x^\epsilon &= v_x^{(1)}(x-ct,y,z,\epsilon t) + \epsilon v_x^{(2)} \\ v_y^\epsilon &= \epsilon^{1/2}\, v_y^{(1)}(x-ct,y,z,\epsilon t) + \epsilon v_y^{(2)} \\ v_z^\epsilon &= \epsilon^{1/2}\, v_z^{(1)}(x-ct,y,z,\epsilon t) + \epsilon v_z^{(2)}, \end{aligned} \tag{74}$$

where the velocity c has to be determined.
Following the strategy of §3.2, we plug this ansatz into (66), and choose the profiles in (74) in order to cancel the leading order terms.

4.2.1. *Cancellation of terms of order ϵ^0.* Canceling the leading order $O(1)$ yields

$$\begin{aligned} -c\partial_X n^{(1)} + \partial_X v_x^{(1)} &= 0, \\ -c\partial_X v_x^{(1)} + \partial_X \phi^{(1)} + \alpha \partial_X n^{(1)} &= 0, \\ (1+\alpha)\partial_y n^{(1)} - a v_z^{(1)} &= 0, \\ (1+\alpha)\partial_z n^{(1)} + a v_y^{(1)} &= 0. \end{aligned} \tag{75}$$

4.2.2. *Cancellation of terms of order $\epsilon^{1/2}$.* We get at this step

$$v_y^{(2)} = \frac{(1+\alpha)^{3/2}}{a^2}\partial^2_{Xy} n^{(1)}, \qquad v_z^{(2)} = \frac{(1+\alpha)^{3/2}}{a^2}\partial^2_{Xz} n^{(1)}. \tag{76}$$

4.2.3. *Cancellation of terms of order ϵ.* Proceeding as in §3.2.3, we get

$$\partial_T n^{(1)} + 2n^{(1)}\partial_X n^{(1)} + \frac{1}{a^2}\partial_X \Delta n^{(1)} = -\partial_X (v_x^{(2)} - cn^{(2)}) \tag{77}$$

$$\partial_X (c v_x^{(2)} - \alpha n^{(2)} - \phi^{(2)}) = c\partial_T n^{(1)} + n^{(1)}\partial_X n^{(1)} \tag{78}$$

$$\partial_y \phi^{(2)} = \frac{(1+\alpha)^{3/2}}{a^2}\partial^3_{XXy} n^{(1)} \tag{79}$$

$$\partial_z \phi^{(2)} = \frac{(1+\alpha)^{3/2}}{a^2}\partial^3_{XXz} n^{(1)} \tag{80}$$

$$\phi^{(1)} = n^{(1)} \tag{81}$$

After replacing $\phi^{(1)}$ by $n^{(1)}$ according to (81), one readily checks that the first two equations of (75) are consistent if and only if $c = \sqrt{1+\alpha}$.

4.2.4. *Cancellation of terms of order $\epsilon^{3/2}$.* As in §(3.2.4), the cancellation of the $O(\epsilon^{3/2})$ terms for the density and the longitudinal velocity equations is automatic, but it is not possible for the equations on the transverse velocity.

4.2.5. *Cancellation of terms of order ϵ^2.* As in §3.2.5 we only need to cancel the $O(\epsilon^2)$ terms in the equation for ϕ^ϵ, which yields here

$$\phi^{(2)} - n^{(2)} = \Delta n^{(1)} - \frac{1}{2}(n^{(1)})^2. \tag{82}$$

4.2.6. *Derivation of the Zakharov-Kuznetsov equation.* Combining (77), (78) and (82), we find that $n^{(1)}$ must solve the following Zakharov-Kuznetsov equation (which coincides with (60) when $\alpha = 0$),

$$2c\partial_T n^{(1)} + 2cn^{(1)}\partial_X n^{(1)} + (1 + \frac{c}{a^2})\Delta\partial_X n^{(1)} = 0. \tag{83}$$

4.2.7. *Construction of the profiles.* The profiles involved in (74) are constructed in terms of $n^{(1)}$ as follows:

- In agreement with (81) and the first two equations of (75), we set $\phi^{(1)} = n^{(1)}$ and $v_x^{(1)} = cn^{(1)}$, with $c = \sqrt{1+\alpha}$.
- The last two equations of (75) then give $v_y^{(1)}, v_z^{(1)}$.
- We then use (76) to obtain $v_y^{(2)}, v_z^{(2)}$.
- We take $\phi^{(2)} = \frac{(1+\alpha)^{3/2}}{a^2}\partial_{XX}^2 n^{(1)}$ to satisfy (79) and (80).
- We then get the density corrector $n^{(2)}$ by (82).
- We recover $v_x^{(2)}$ from (77) or (78) – this is equivalent since $n^{(1)}$ solves the ZK equation (83).

Finally we get the following consistency result that generalized Proposition 2 when isothermal pressure is taken into account.

Proposition 3. *Let $T_0 > 0$, $n_0 \in H^s(\mathbb{R}^d)$ $(s \geq 5)$ and $n^{(1)} \in C([0,T_0]; H^s(\mathbb{R}^d))$, solving (with $c = \sqrt{1+\alpha}$),*

$$2c\partial_T n^{(1)} + 2cn^{(1)}\partial_X n^{(1)} + \Big(1 + \frac{c}{a^2}\Big)\Delta\partial_X n^{(1)} = 0, \qquad n^{(1)}_{|t=0} = n_0.$$

Constructing the other profiles as indicated above, the approximate solution $(n^\epsilon, \mathbf{v}^\epsilon, \phi^\epsilon)$ given by (74) *solves (66) up to order ϵ^3 in ϕ^ϵ, ϵ^2 in n^ϵ, v_x^ϵ, and up to order $\epsilon^{3/2}$ in v_y^ϵ, v_z^ϵ:*

$$\begin{cases} \partial_t n^\epsilon + \nabla\cdot((1+\epsilon n^\epsilon)\mathbf{v}^\epsilon) = \epsilon^2 N^\epsilon, \\ \partial_t \mathbf{v}^\epsilon + \epsilon(\mathbf{v}^\epsilon\cdot\nabla)\mathbf{v}^\epsilon + \nabla\phi^\epsilon + \alpha\dfrac{\nabla n}{1+\epsilon n} + a\epsilon^{-1/2}\mathbf{e}\wedge\mathbf{v}^\epsilon = \epsilon^{3/2}R^\epsilon, \\ -\epsilon^2\Delta\phi^\epsilon + e^{\epsilon\phi^\epsilon} - 1 = \epsilon n^\epsilon + \epsilon^3 r^\epsilon, \end{cases} \tag{84}$$

with $R^\epsilon = (\epsilon^{1/2}R_1^\epsilon, R_2^\epsilon, R_3^\epsilon)$ and

$$|N^\epsilon|_{L^\infty([0,\frac{T_0}{\epsilon}];H^{s-5})} + \sum_{j=1}^3 |R_j^\epsilon|_{L^\infty([0,\frac{T_0}{\epsilon}];H^{s-5})} + |r^\epsilon|_{L^\infty([0,\frac{T_0}{\epsilon}];H^{s-4})} \leq C\big(T_0, |n_0|_{H^s}\big).$$

4.3. **Justification of the Zakharov-Kuznetsov approximation.** Proceeding exactly as for Theorem 3 but replacing Theorem 2 by Theorem 4, we get the following justification of the Zakharov-Kuznetsov approximation in presence of a isothermal pressure.

Theorem 5. *Let $n^0 \in H^{s+5}$, with $s > d/2+1$, such that $1 + \epsilon n^0 \geq c_0$ on $\mathbb{R}^d$ for some constant $c_0 > 0$. There exists $T_1 > 0$ such that*
i. *The Zakharov-Kuznetsov approximation $(n^\epsilon, \mathbf{v}^\epsilon, \phi^\epsilon)$ of Proposition 3 exists on the time interval $[0, T_1/\epsilon]$;*
ii. *There exists a unique solution $(\underline{n}, \underline{\mathbf{v}}, \underline{\phi}) \in C([0, \frac{T_1}{\epsilon}]; H^{s+1}_{\epsilon\alpha}(\mathbb{R}^d) \times H^{s+1}_\epsilon(\mathbb{R}^d)^d \times H^{s+1}(\mathbb{R}^d))$ provided by Theorem 4 to the Euler-Poisson equations with isothermal pressure* (66) *with initial condition $(\underline{n}^0, \underline{\mathbf{v}}^0, \underline{\phi}^0) = (n^\epsilon, \mathbf{v}^\epsilon, \phi^\epsilon)_{|t=0}$.*

Moreover, one has the error estimate

$$\forall 0\leq t\leq T_1/\epsilon,\qquad |\underline{n}(t)-n^\epsilon(t)|^2_{H^{s+1}_{\epsilon\alpha}}+|\underline{\mathbf{v}}(t)-\mathbf{v}^\epsilon(t)|^2_{H^{s+1}_\epsilon}\leq \epsilon^{3/2}tC(\frac{1}{c_0},T_1,|n^0|_{H^{s+5}}).$$

Remark 8. The comments made in Remarks 4 and 5 on the precision of the Zakharov-Kuznetsov approximation for cold plasmas can be transposed to the more general case considered here.

Remark 9. As already said, the ZK equation (83) coincides in dimension $d=1$ with the KdV equation derived in [11]. A consequence of the uniformity of the existence time with respect to α in Theorem 4 is that Theorem 5 provides a justification on a time scale of order $O(1/\epsilon)$ which is uniform with respect to α whereas it shrinks to zero when $\alpha\to 0$ in [11].

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